Least-Squares Rigid Motion Using SVD

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Abstract

This note summarizes the steps to computing the rigid transformation that aligns two sets of points.

Key words: Shape matching, rigid alignment, rotation, SVD

1 Problem statement

Let $P = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ and $Q = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$ be two sets of corresponding points in $\mathbb{R}^d$. We wish to find a rigid transformation that optimally aligns the two sets in the least squares sense, i.e., we seek a rotation R and a translation vector $\mathbf{t}$ such that

$$(R, \mathbf{t}) = \underset{R,\mathbf{t}}{\operatorname{argmin}} \sum_{i=1}^{n} w_i \left\| (R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i \right\|^2, \tag{1}$$

where $w_i > 0$ are weights for each point pair.

In the following we will detail the derivation of R and $\mathbf{t}$; readers that are interested in the final recipe may skip the proofs and go directly Section 4.

2 Computing the translation

Assume R is fixed and denote $F(\mathbf{t}) = \sum_{i=1}^{n} w_i \left\| (R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i \right\|^2$. We can find the optimal translation by taking the derivative of F w.r.t. $\mathbf{t}$ and searching for its roots:

$$\begin{aligned} 0 = \frac{\partial F}{\partial \mathbf{t}} &= \sum_{i=1}^{n} 2w_i \left(R\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right) = \\ &= 2\mathbf{t} \left(\sum_{i=1}^{n} w_i \right) + 2R \left(\sum_{i=1}^{n} w_i \mathbf{p}_i \right) - 2 \sum_{i=1}^{n} w_i \mathbf{q}_i. \end{aligned} \tag{2}$$

Denote

$$\bar{\mathbf{p}} = \frac{\sum_{i=1}^n w_i \mathbf{p}_i}{\sum_{i=1}^n w_i}, \quad \bar{\mathbf{q}} = \frac{\sum_{i=1}^n w_i \mathbf{q}_i}{\sum_{i=1}^n w_i}. \tag{3}$$

By rearranging the terms of (2) we get

$$\mathbf{t} = \bar{\mathbf{q}} - R\bar{\mathbf{p}}. \tag{4}$$

In other words, the optimal translation $\mathbf{t}$ maps the transformed weighted centroid of P to the weighted centroid of Q. Let us plug the optimal $\mathbf{t}$ into our objective function:

$$\sum_{i=1}^n w_i \left\|(R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\right\|^2 = \sum_{i=1}^n w_i \left\|R\mathbf{p}_i + \bar{\mathbf{q}} - R\bar{\mathbf{p}} - \mathbf{q}_i\right\|^2 = \tag{5}$$

$$= \sum_{i=1}^n w_i \left\|R(\mathbf{p}_i - \bar{\mathbf{p}}) - (\mathbf{q}_i - \bar{\mathbf{q}})\right\|^2. \tag{6}$$

We can thus concentrate on computing the rotation R by restating the problem such that the translation would be zero:

$$\mathbf{x}_i := \mathbf{p}_i - \bar{\mathbf{p}}, \quad \mathbf{y}_i := \mathbf{q}_i - \bar{\mathbf{q}}. \tag{7}$$

So we look for the optimal rotation R such that

$$R = \operatorname*{argmin}_R \sum_{i=1}^n w_i \left\|R\mathbf{x}_i - \mathbf{y}_i\right\|^2. \tag{8}$$

3 Computing the rotation

Let us simplify the expression we are trying to minimize in (8):

$$\begin{aligned} \left\|R\mathbf{x}_i - \mathbf{y}_i\right\|^2 &= (R\mathbf{x}_i - \mathbf{y}_i)^T(R\mathbf{x}_i - \mathbf{y}_i) = (\mathbf{x}_i^T R^T - \mathbf{y}_i^T)(R\mathbf{x}_i - \mathbf{y}_i) = \\ &= \mathbf{x}_i^T R^T R\mathbf{x}_i - \mathbf{y}_i^T R\mathbf{x}_i - \mathbf{x}_i^T R^T \mathbf{y}_i + \mathbf{y}_i^T \mathbf{y}_i = \\ &= \mathbf{x}_i^T \mathbf{x}_i - \mathbf{y}_i^T R\mathbf{x}_i - \mathbf{x}_i^T R^T \mathbf{y}_i + \mathbf{y}_i^T \mathbf{y}_i. \end{aligned} \tag{9}$$

We got the last step by remembering that rotation matrices imply $R^T R = I$ (I is the identity matrix).

Note that $\mathbf{x}_i^T R^T \mathbf{y}_i$ is a scalar: $\mathbf{x}_i^T$ has dimension $1 \times d$, R^T is $d \times d$ and $\mathbf{y}_i$ is $d \times 1$. For any scalar a we trivially have $a = a^T$, therefore

$$\mathbf{x}_i^T R^T \mathbf{y}_i = (\mathbf{x}_i^T R^T \mathbf{y}_i)^T = \mathbf{y}_i^T R\mathbf{x}_i. \tag{10}$$

Therefore we have

$$\|R\mathbf{x}_i - \mathbf{y}_i\|^2 = \mathbf{x}_i^T\mathbf{x}_i - 2\mathbf{y}_i^T R\mathbf{x}_i + \mathbf{y}_i^T\mathbf{y}_i. \tag{11}$$

Let us look at the minimization and substitute the above expression:

$$\begin{aligned}
&\underset{R}{\operatorname{argmin}} \sum_{i=1}^{n} w_i \|R\mathbf{x}_i - \mathbf{y}_i\|^2 = \underset{R}{\operatorname{argmin}} \sum_{i=1}^{n} w_i(\mathbf{x}_i^T\mathbf{x}_i - 2\mathbf{y}_i^T R\mathbf{x}_i + \mathbf{y}_i^T\mathbf{y}_i) = \\
&= \underset{R}{\operatorname{argmin}} \left(\sum_{i=1}^{n} w_i\mathbf{x}_i^T\mathbf{x}_i - 2\sum_{i=1}^{n} w_i\mathbf{y}_i^T R\mathbf{x}_i + \sum_{i=1}^{n} w_i\mathbf{y}_i^T\mathbf{y}_i\right) = \\
&= \underset{R}{\operatorname{argmin}} \left(-2\sum_{i=1}^{n} w_i\mathbf{y}_i^T R\mathbf{x}_i\right).
\end{aligned} \tag{12}$$

The last step (removing $\sum_{i=1}^{n} w_i\mathbf{x}_i^T\mathbf{x}_i$ and $\sum_{i=1}^{n} w_i\mathbf{y}_i^T\mathbf{y}_i$) holds because these expressions do not depend on R at all, so excluding them would not affect the minimizer. The same holds for multiplication of the minimization expression by a scalar, so we have

$$\underset{R}{\operatorname{argmin}} \left(-2\sum_{i=1}^{n} w_i\mathbf{y}_i^T R\mathbf{x}_i\right) = \underset{R}{\operatorname{argmax}} \sum_{i=1}^{n} w_i\mathbf{y}_i^T R\mathbf{x}_i. \tag{13}$$

We note that

$$\sum_{i=1}^{n} w_i\mathbf{y}_i^T R\mathbf{x}_i = tr\left(WY^T RX\right), \tag{14}$$

where $W = \operatorname{diag}(w_1, \ldots, w_n)$ is an $n \times n$ diagonal matrix with the weight w_i on diagonal entry i; Y is the $d \times n$ matrix with $\mathbf{y}_i$ as its columns and X is the $d \times n$ matrix with $\mathbf{x}_i$ as its columns. We remind the reader that the trace of a square matrix is the sum of the elements on the diagonal: $tr(A) = \sum_{i=1}^{n} a_{ii}$. See Figure 1 for an illustration of the algebraic manipulation.

Therefore we are looking for a rotation R that maximizes $tr\left(WY^T RX\right)$. Matrix trace has the property

$$tr(AB) = tr(BA) \tag{15}$$

for any matrices A, B of compatible dimensions. Therefore

$$tr\left(WY^T RX\right) = tr\left((WY^T)(RX)\right) = tr\left(RXWY^T\right). \tag{16}$$

Let us denote the $d \times d$ "covariance" matrix $S = XWY^T$. Take SVD of S:

$$S = U\Sigma V^T. \tag{17}$$

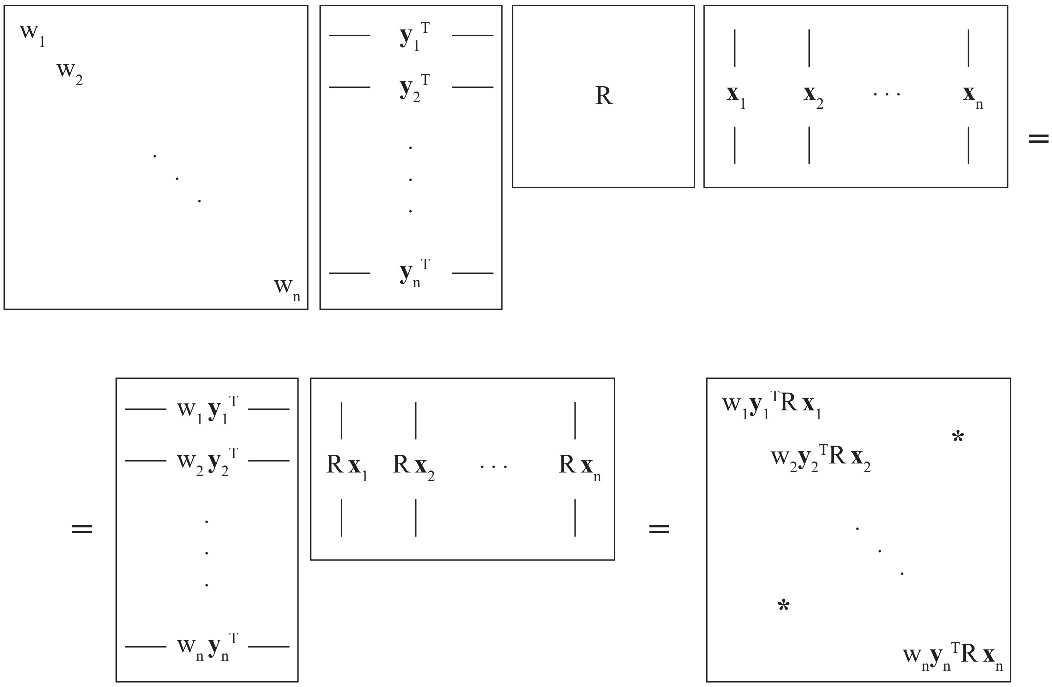

Fig. 1. Schematic explanation of $\sum_{i=1}^{n} w_i \mathbf{y}_i^T R\mathbf{x}_i = tr(WY^TRX)$.

Now substitute the decomposition into the trace we are trying to maximize:

$$tr\left(RXWY^T\right) = tr\left(RS\right) = tr\left(RU\Sigma V^T\right) = tr\left(\Sigma V^T RU\right). \tag{18}$$

The last step was achieved using the property of trace (15). Note that V, R and U are all orthogonal matrices, so $M = V^T RU$ is also an orthogonal matrix. This means that the columns of M are orthonormal vectors, and in particular, $\mathbf{m}_j^T \mathbf{m}_j = 1$ for each M's column $\mathbf{m}_j$. Therefore all entries m_{ij} of M are smaller than 1 in magnitude:

$$1 = \mathbf{m}_j^T \mathbf{m}_j = \sum_{i=1}^{d} m_{ij}^2 \;\Rightarrow\; m_{ij} \leq 1 \;\Rightarrow\; |m_{ij}| < 1\,. \tag{19}$$

So what is the maximum possible value for $tr(\Sigma M)$? Remember that Σ is a diagonal matrix with non-negative values $\sigma_1, \sigma_2, \ldots, \sigma_d \geq 0$ on the diagonal. Therefore:

$$tr(\Sigma M) = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1d} \\ m_{21} & m_{22} & \ldots & m_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ m_{d1} & m_{d2} & \ldots & m_{dd} \end{pmatrix} = \sum_{i=1}^{d} \sigma_i m_{ii} \leq \sum_{i=1}^{d} \sigma_i. \tag{20}$$

Therefore the trace is maximized if $m_{ii} = 1$. Since M is an orthogonal matrix, this means that M would have to be the identity matrix!

$$I = M = V^T RU \;\Rightarrow\; V = RU \;\Rightarrow\; R = VU^T. \tag{21}$$

Orientation rectification. The process we just described finds the optimal *orthogonal* matrix, which could potentially contain reflections in addition to rotations. Imagine that the point set P is a perfect reflection of Q – we will then find that reflection, which aligns the two point sets perfectly and yields zero energy (8) – the global minimum in this case. However, if we restrict ourselves to rotations only, there might not be a rotation that perfectly aligns the points.

Checking whether $R = VU^T$ is a rotation is simple: if $\det(VU^T) = -1$ it contains reflection, otherwise $\det(VU^T) = +1$. Assume $\det(VU^T) = -1$: this means that the global maximum of $tr(\Sigma M)$ is generally not attainable by a rotation, and we need to look for the "next best thing". Let us look for other (local) maxima of $tr(\Sigma M)$ as a function of M's diagonal values m_{ii}:

$$tr(\Sigma M) = \sigma_1 m_{11} + \sigma_2 m_{22} + \ldots + \sigma_d m_{dd} =: f(m_{11}, \ldots, m_{dd}). \tag{22}$$

If we consider the m_{ii}'s as variables, the domain of $(m_{11}, \ldots, m_{dd})$ is a subset of $[-1, 1]^d$. The function f is linear in the m_{ii}'s, so it attains its extrema on the boundary of the domain (no extrema on the interior). Since our domain is rectilinear, the extrema will be attained at the vertices $(\pm 1, \pm 1, \ldots, \pm 1)$. We had to rule out $(1, 1, \ldots, 1)$ since that gave a reflection, therefore the next best shot is $(1, 1, \ldots, 1, -1)$:

$$tr(\Sigma M) = \sigma_1 + \sigma_2 + \ldots + \sigma_{d-1} - \sigma_d. \tag{23}$$

This is larger than any other combination (except $(1, 1, \ldots, 1)$) because σ_d is the smallest singular value.

To summarize, we arrive at the fact that if $\det(VU^T) = -1$, we need

$$M = V^T R U = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \quad \Rightarrow \quad R = V \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} U^T. \tag{24}$$

We can write a general formula that encompasses both cases, $\det(VU^T) = 1$ and $\det(VU^T) = -1$:

$$R = V \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \det(VU^T) \end{pmatrix} U^T. \tag{25}$$

4 Rigid motion computation – summary

Let us summarize the steps to computing the optimal translation $\mathbf{t}$ and rotation R that minimize

$$\sum_{i=1}^{n} w_i \left\|(R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\right\|^2 .$$

(1) Compute the weighted centroids of both point sets:

$$\bar{\mathbf{p}} = \frac{\sum_{i=1}^{n} w_i \mathbf{p}_i}{\sum_{i=1}^{n} w_i}, \quad \bar{\mathbf{q}} = \frac{\sum_{i=1}^{n} w_i \mathbf{q}_i}{\sum_{i=1}^{n} w_i} .$$

(2) Compute the centered vectors

$$\mathbf{x}_i := \mathbf{p}_i - \bar{\mathbf{p}}, \quad \mathbf{y}_i := \mathbf{q}_i - \bar{\mathbf{q}}, \qquad i = 1, 2, \ldots, n.$$

(3) Compute the $d \times d$ covariance matrix

$$S = XWY^T,$$

where X and Y are the $d \times n$ matrices that have $\mathbf{x}_i$ and $\mathbf{y}_i$ as their columns, respectively, and $W = \mathrm{diag}(w_1, w_2, \ldots, w_n)$.

(4) Compute the singular value decomposition $S = U\Sigma V^T$. The rotation we are looking for is then

$$R = V \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \det(VU^T) \end{pmatrix} U^T.$$

(5) Compute the optimal translation as

$$\mathbf{t} = \bar{\mathbf{q}} - R\bar{\mathbf{p}}.$$