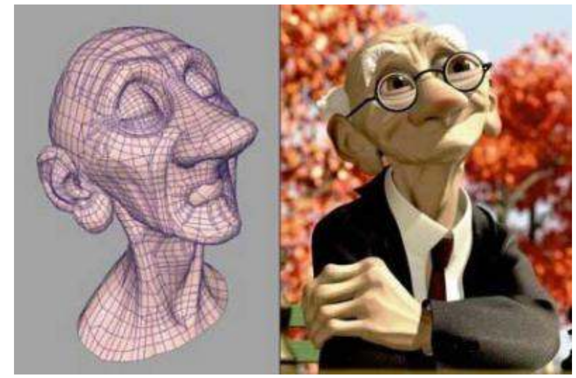
CS 523: Computer Graphics, Spring 2011

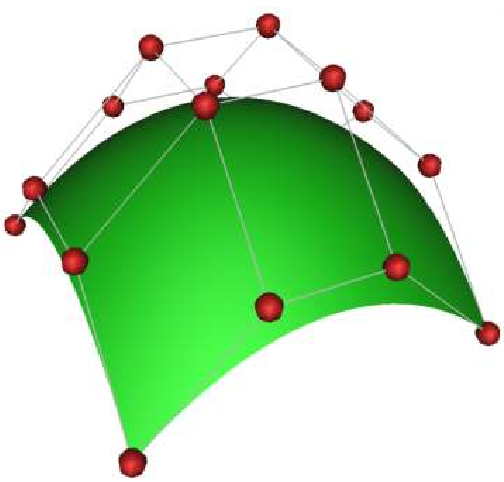
Shape Modeling

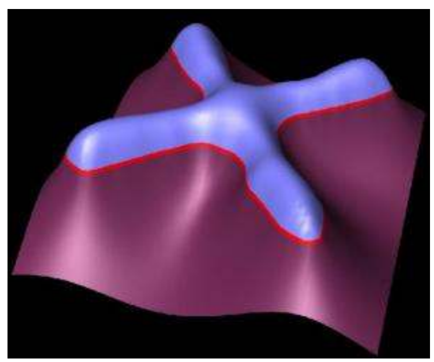
Shape Representations

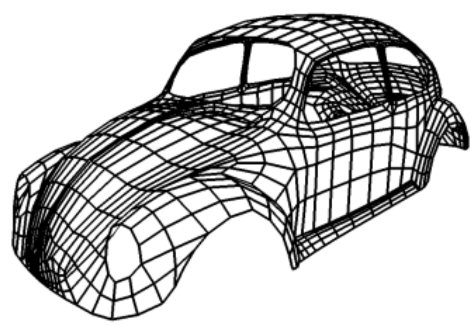
Course topics

- Shape representation
  - Points
  - Parametric surfaces
  - Implicits



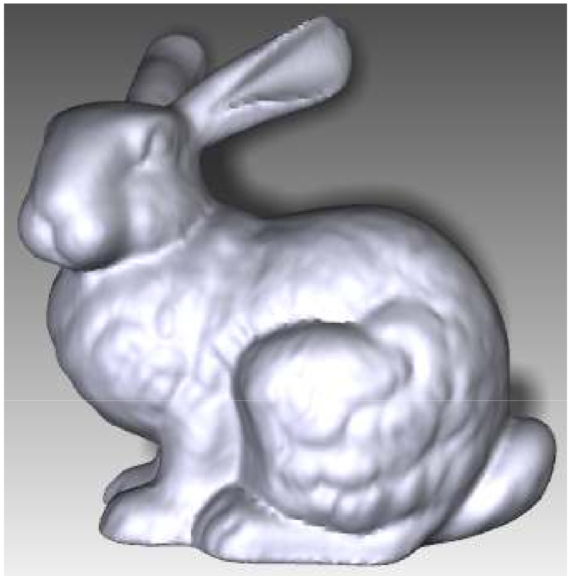


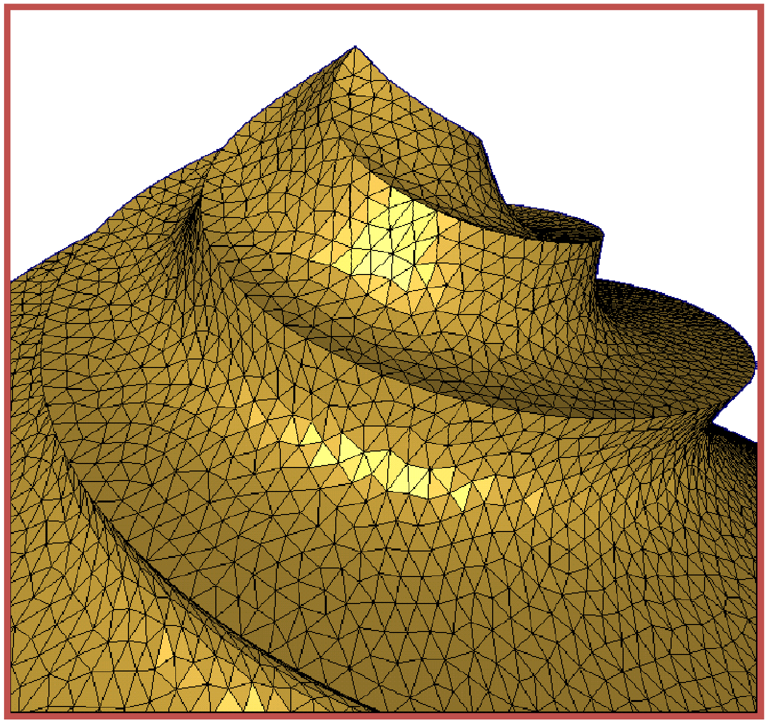


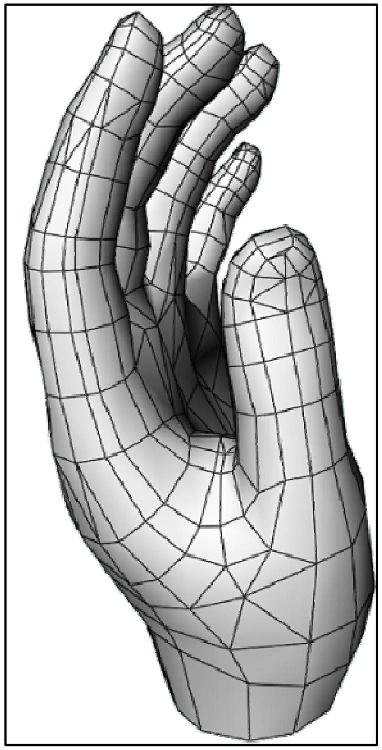


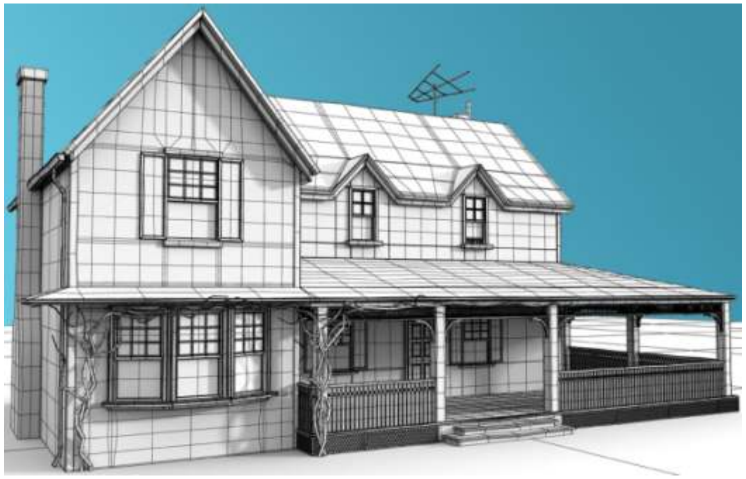
Course topics

- Shape representation
  - Subdivision surfaces
  - Polygonal meshes



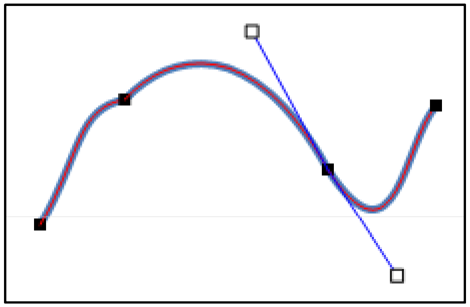


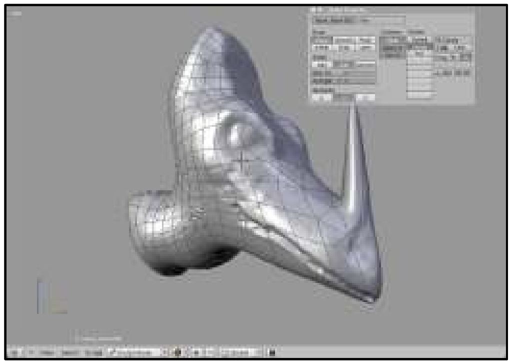




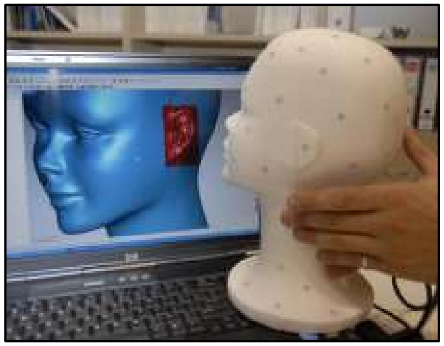
Shape representation

- Where does the shape come from?
- Modeling "by hand":
  - Higher-level representations, amendable to modification, control
  - Parametric surfaces, subdivision surfaces, implicits





- Acquired real-world objects:
  - Discrete sampling
  - Points, meshes



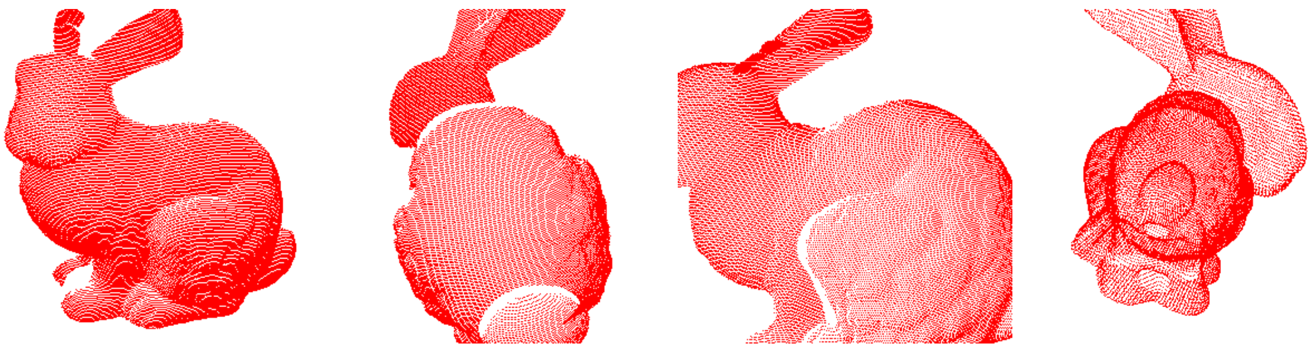
Points

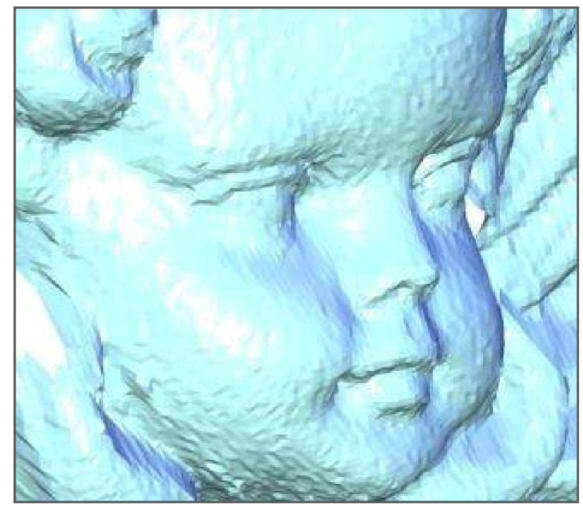
Shape acquisition

Sampling of real world objects

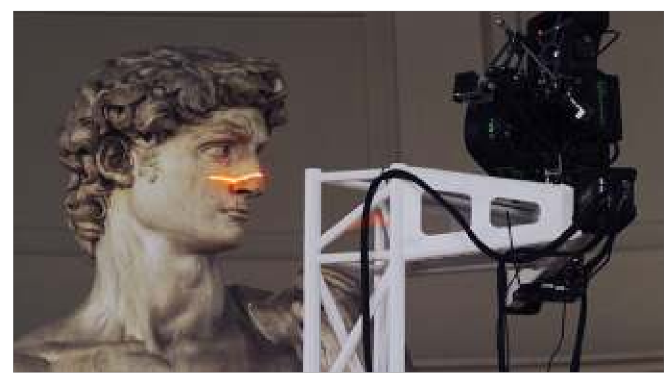
Points

- Standard 3D data from a variety of sources
  - Often results from scanners
  - Potentially noisy



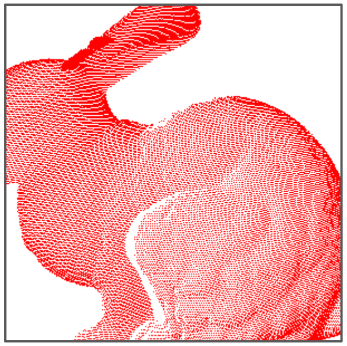


  - Depth imaging (e.g. by triangulation
  - Registration of multiple images



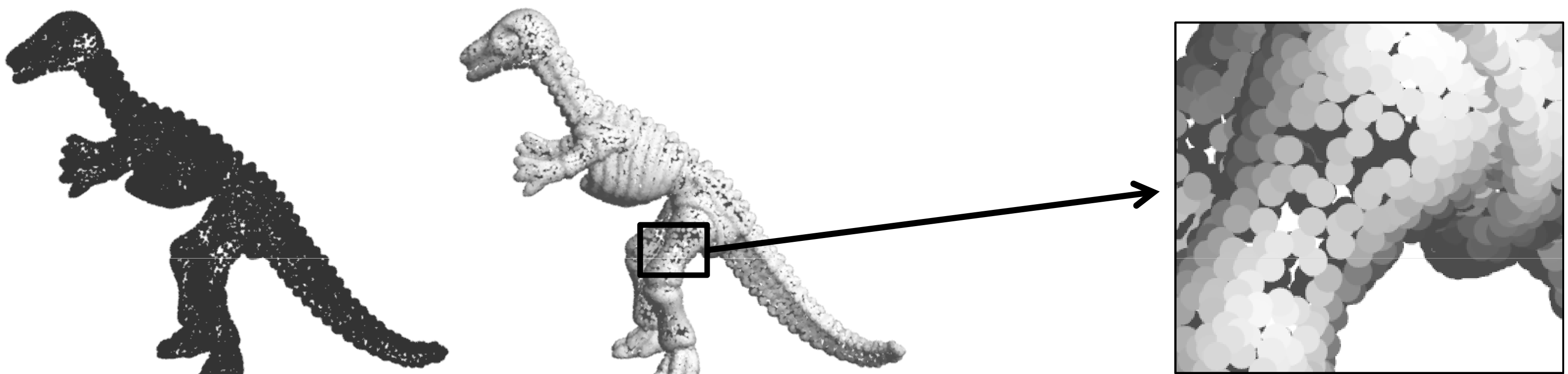
Points

- points = unordered set of 3-tuples
- Often converted to other reps
  - Meshes, implicits, parametric surfaces
  - Easier to process, edit and/or render
- Efficient point processing and modeling requires a spatial partitioning data structure
  - To figure out neighborhoods



Points
Neighborhood information

- Why do we need neighbors?



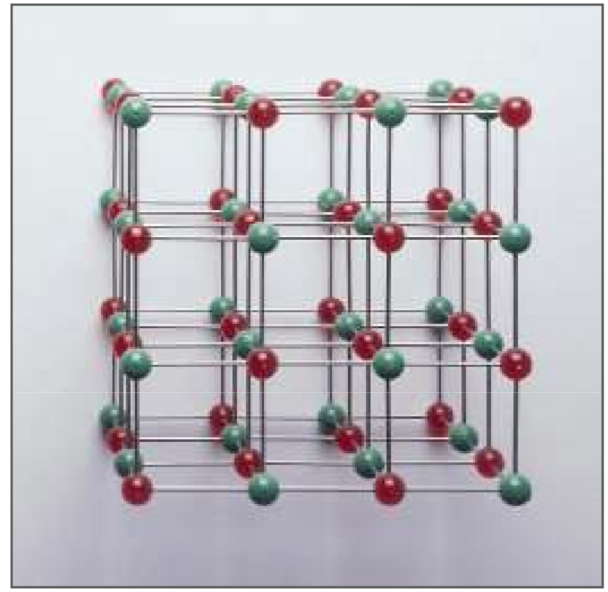
need normals (for shading) upsampling – need to count density

- Need sub-linear implementations of
  - k-nearest neighbors to point $\mathbf{x}$
  - In radius search $\|\mathbf{p}_i - \mathbf{x}\| < \varepsilon$

Spatial Data Structures

Commonly used for point processing

- Regular uniform 3D lattice
  - Simple point insertion by coordinate discretization
  - Simple proximity queries by searching neighboring cells

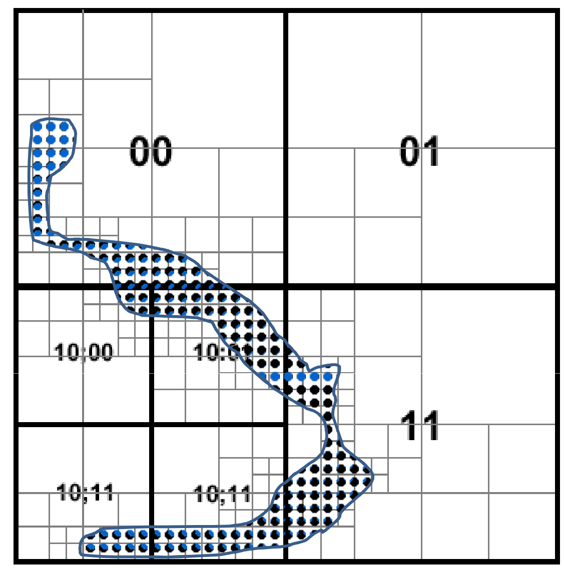


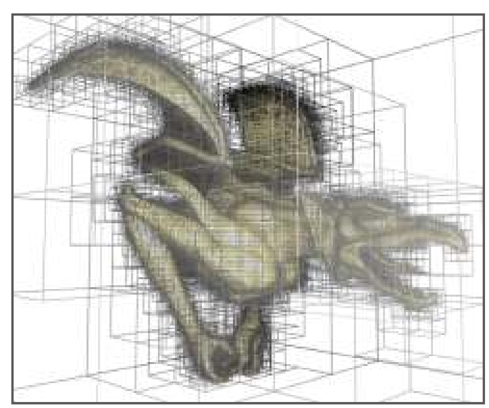
  - Determining lattice parameters (i.e. cell dimensions) is nontrivial
  - Generally unbalanced, i.e. many empty cells

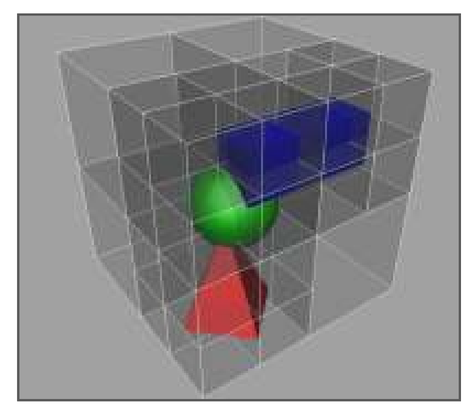
Spatial Data Structures

Commonly used for point processing

- Octree
  - Splits each cell into 8 equal cells
  - Adaptive, i.e. only splits when too many points in cell
  - Proximity search by (recursive) tree traversal and distance to neighboring cells
  - Tree might not be balanced



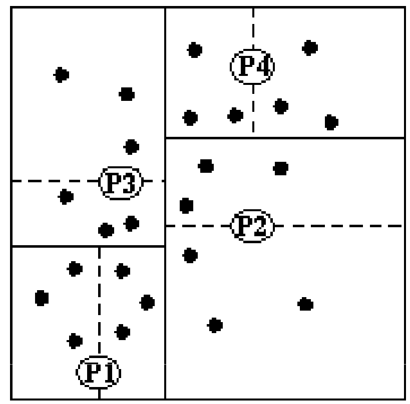


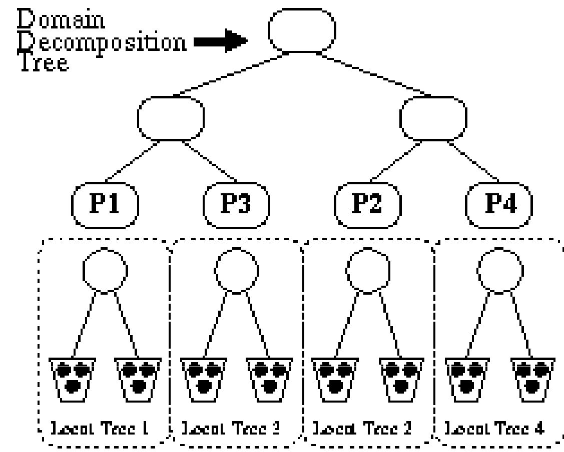


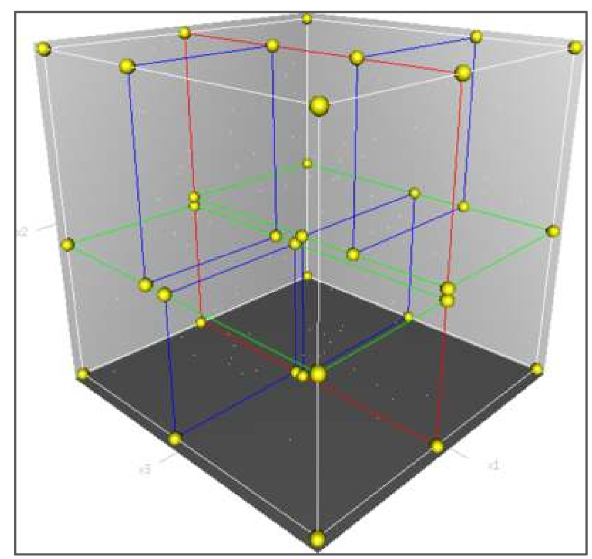
Spatial Data Structures

Commonly used for point processing

- Kd-Tree
  - Each cell is individually split along the median into two cells
  - Same amount of points in cells
  - Perfectly balanced tree
  - Proximity search similar to the recursive search in an Octree
  - More data storage required for inhomogeneous cell dimensions



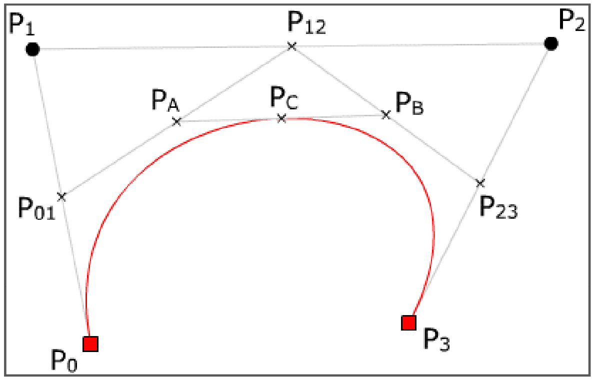




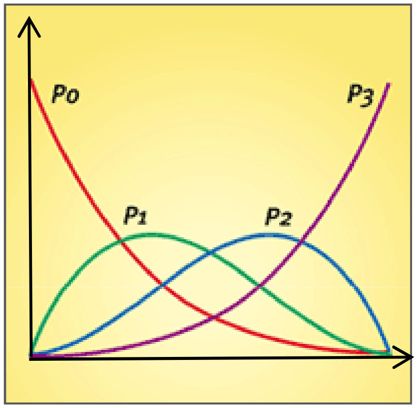
Parametric Curves and Surfaces

Parametric Curves and Surfaces

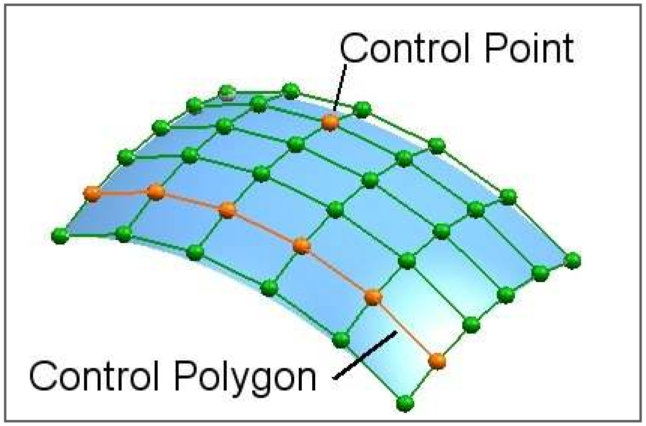
- Curves are 1-dimensional parameterizations



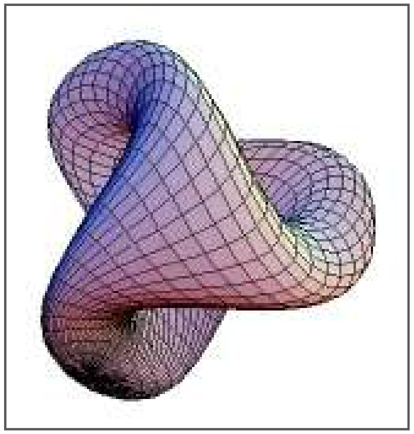
$$S(t) = \mathbf{x}_t$$



- Surfaces are 2-dimensional parameterizations



$$S(x, y) = \mathbf{x}_t$$



Parametric Curves and Surfaces

Examples

- Explicit curve/circle in 2D

$$\mathbf{p} : R \rightarrow R^d, d = 1,2,3,\ldots$$

$$t \mapsto \mathbf{p}(t) = \left(x(t), y(t), z(t)\right)$$

$$\mathbf{p}(t) = r \cdot \left(\cos(t), \sin(t), 0\right)$$

$$t \in [0, 2\pi]$$

Parametric Curves and Surfaces

Examples

- Explicit surface/sphere in 3D

$$\mathbf{q} : R^2 \to R^d, d = 1,2,3,\ldots$$

$$(u,v) \mapsto \mathbf{q}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

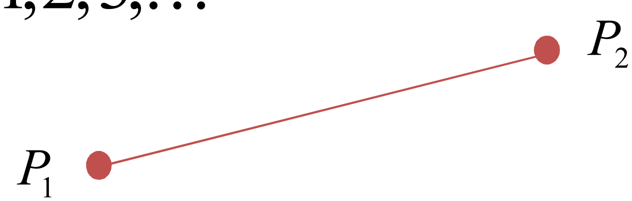
$$\mathbf{p}(u,v) = r \cdot \big(\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)\big)$$

$$(u,v) \in [0,2\pi] \times [-\pi/2, \pi/2]$$

Parametric Curves

Continuity and regularity

- Curve segment $\mathbf{p}:[a,b] \to R^d, d = 1,2,3,\ldots$
  - The same segment can be parameterized differently



$$\mathbf{p}_1 : [0,1] \to R^3, \mathbf{p}(t) = tP_1 + (1-t)P_2$$
$$\mathbf{p}_2 : [0,1] \to R^3, \mathbf{p}(t) = t^2 P_1 + (1-t^2)P_2$$

  - A parametric curve is n-times continuously differentiable if the image $\mathbf{p}$ is n-times continuously differentiable (C^n)
  - The derivative $\mathbf{p}'(t)$ at position t is a tangent vector
  - A curve is regular when $\mathbf{p}$ is differentiable and $\mathbf{p}'(t) \neq \mathbf{0}$

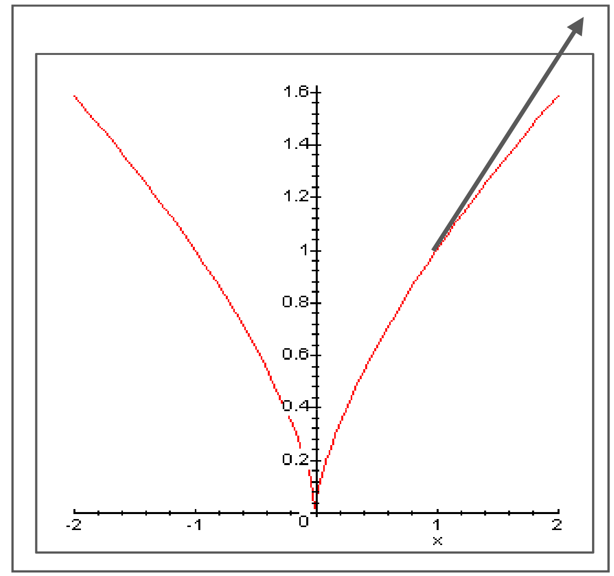
Parametric Curves

Continuity and regularity

- Example

$$\mathbf{p}:[-2,2] \rightarrow R^3, \mathbf{p}(t) = (t^3, t^2, 0)$$

$$\mathbf{p}'(t) = (3t^2, 2t, 0) \Rightarrow \mathbf{p}'(0) = 0$$



  - $\mathbf{p}$ is continuously differentiable, but not regular at position $t = 0$
  - The regularity of a curve can be expressed as its visual smoothness
  - The tangent vector can be interpreted as the velocity (compare to physics $\mathbf{v} = \mathbf{s}'$)

Parametric Curves

Arc length parameterization

- A curve is parameterized by arc length when

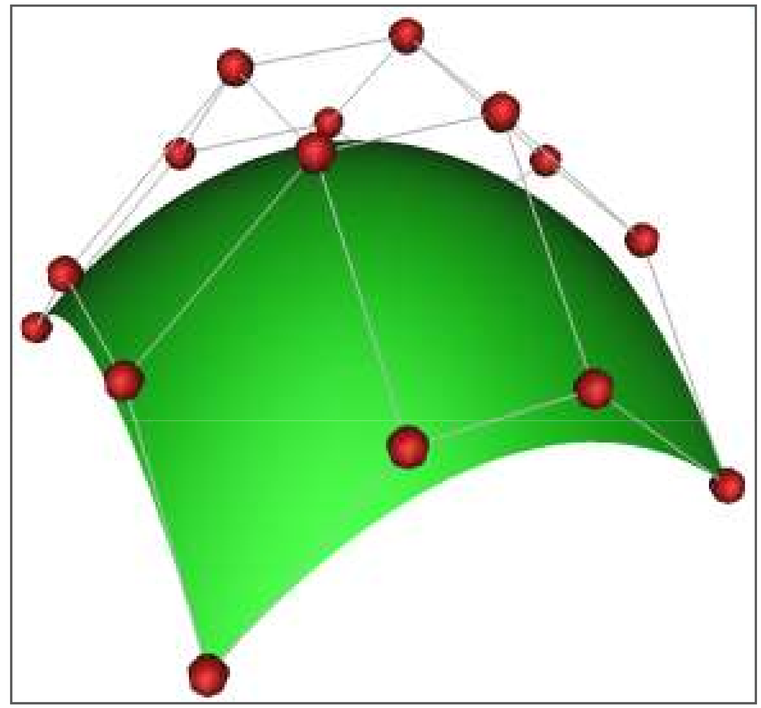
$$\left\|\mathbf{p}'(t)\right\| = 1,\ \ t \in [a,b]$$

  - Any regular curve can be parameterized by arc length
  - For arc length parameterized curves:

$T(s) \coloneqq \mathbf{p}'(s)$	Tangent vector
$K(s) \coloneqq \mathbf{p}''(s)$	Curvature vector
$\kappa(s) \coloneqq \left\|\mathbf{p}''(s)\right\|$	Curvature (scalar)

Parametric Surfaces

Tensor product surfaces



- Example: Bezier surfaces
  - Surface lies in convex hull of control points
  - Surface interpolates the four corner control points
  - Boundary curves are Bezier curves defined only by control points on the boundary
- Other: B-Spline patches, NURBS, etc…

Parametric Curves and Surfaces

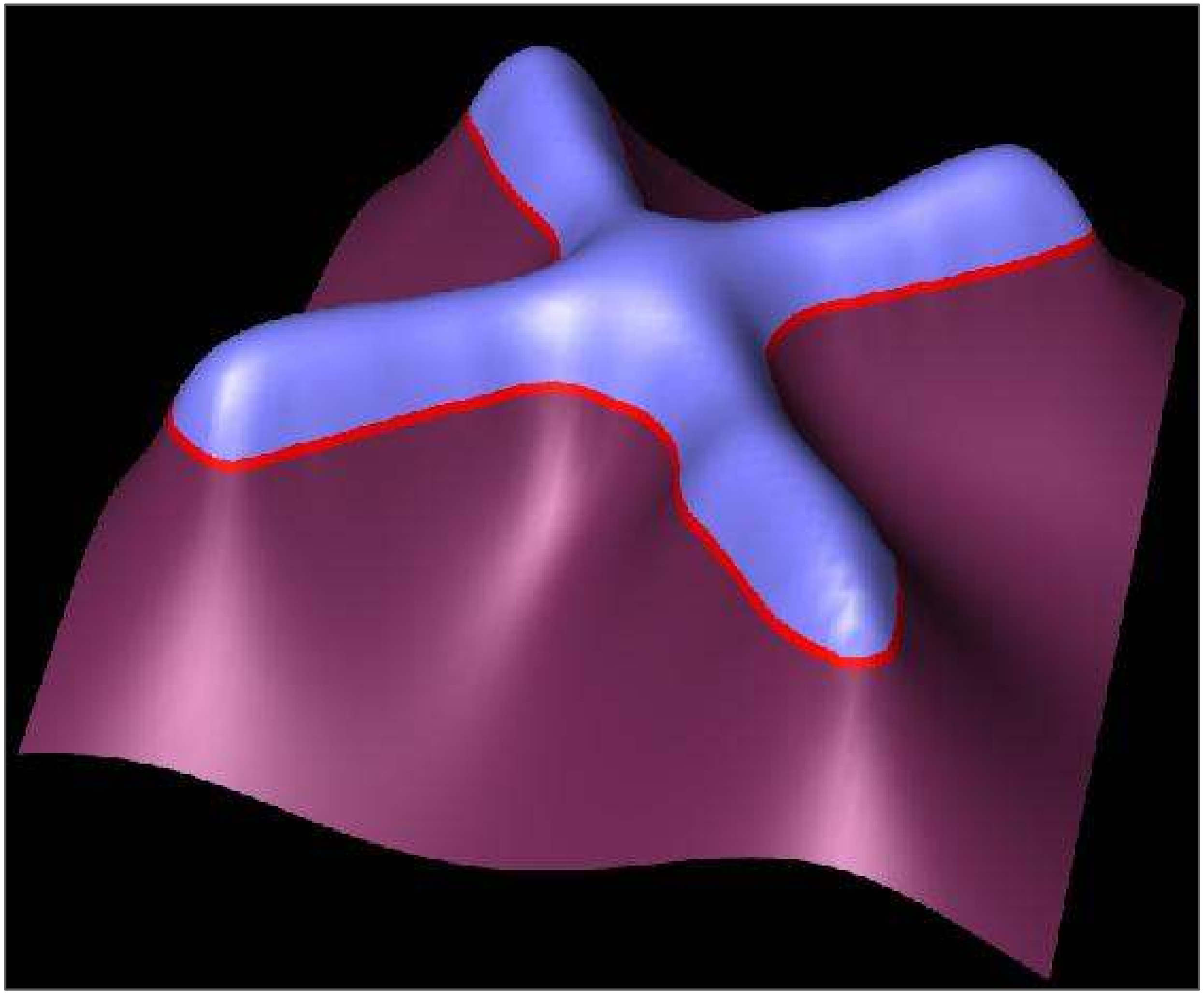
- Advantages
  - Easy to generate points on the curve/surface
  - Separates x/y/z components

- Disadvantages
  - Hard to determine inside/outside
  - Hard to determine if a point is on the curve/surface

Implicit Curves and Surfaces

Implicit Curves and Surfaces

Illustration



Implicit Curves and Surfaces

Examples

- Implicit circle and sphere

$$f : R^2 \rightarrow R \qquad\qquad g : R^3 \rightarrow R$$

$$K = \{\mathbf{p} \in R^2 : f(\mathbf{p}) = 0\} \qquad\qquad K = \{\mathbf{p} \in R^3 : g(\mathbf{p}) = 0\}$$

$$f(x, y) = x^2 + y^2 - r^2 \qquad\qquad g(x, y, z) = x^2 + y^2 + z^2 - r^2$$

Implicit Curves and Surfaces

Definition

- Definition $g : \mathrm{R}^3 \to \mathrm{R}$

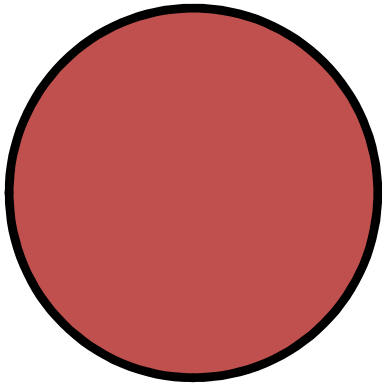
$$K = g^{-1}(0) = \{\mathbf{p} \in \mathrm{R}^3 : g(\mathbf{p}) = 0\}$$

- Space partitioning

$\{\mathbf{p} \in \mathrm{R}^3 : g(\mathbf{p}) < 0\}$ Inside

$\{\mathbf{p} \in \mathrm{R}^3 : g(\mathbf{p}) = 0\}$ Curve/Surface

$\{\mathbf{p} \in \mathrm{R}^3 : g(\mathbf{p}) > 0\}$ Outside



Implicit Curves and Surfaces

Gradient

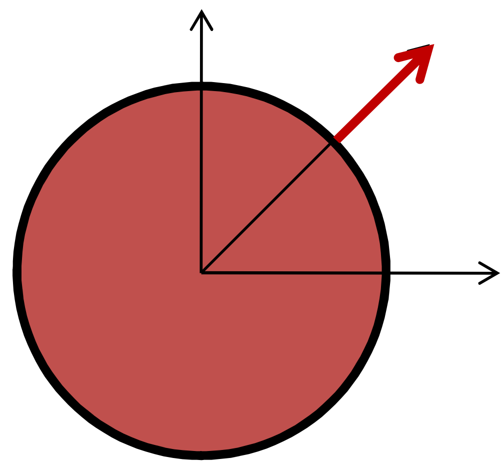
- The normal vector to the surface is given by the gradient of the (scalar) implicit function

$$\nabla g(x,y,z) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right)^{\mathrm{T}}$$

- Example

$$g(x,y,z) = x^2 + y^2 + z^2 - r^2$$

$$\nabla g(x,y,z) = (2x,2y,2z)^{\mathrm{T}}$$

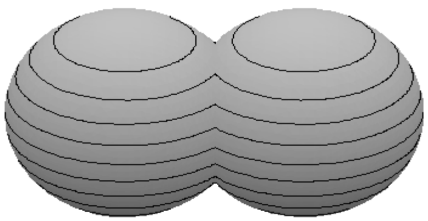


$$\nabla g(x,y,z) = (2,2,0)^{\mathrm{T}}$$

Implicit Curves and Surfaces

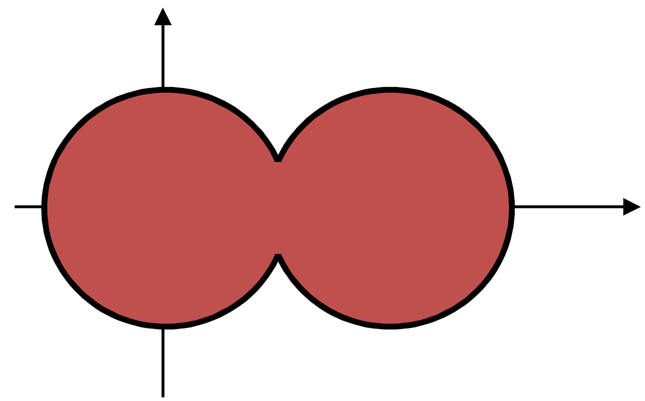
Smooth set operations

- Standard operations: union and intersection



$$\bigcup_i g_i(\mathbf{p}) = \min g_i(\mathbf{p})$$

$$\bigcap_i g_i(\mathbf{p}) = \max g_i(\mathbf{p})$$



- In many cases, smooth blending is desired
  - Pasko and Savchenko [1994]

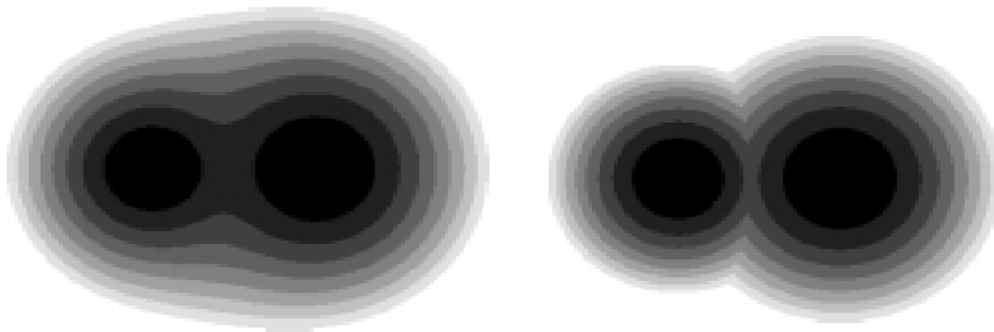
$$g \cup f = \tfrac{1}{1+\alpha}\left(g + f - \sqrt{g^2 + f^2 - 2\alpha g f}\right)$$

$$g \cap f = \tfrac{1}{1+\alpha}\left(g + f + \sqrt{g^2 + f^2 - 2\alpha g f}\right)$$

Implicit Curves and Surfaces

Smooth set operations

- Examples



$\alpha = 0$ $\alpha = 1$

- For $\alpha = 1$, this is equivalent to $\min$ and $\max$

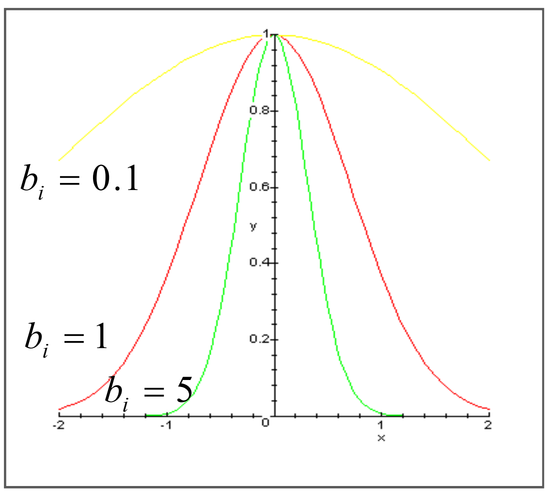
$$\lim_{\alpha \to 1} g \cup f = \tfrac{1}{2}\left(g + f - \sqrt{(g-f)^2}\right) = \frac{g+f}{2} - \frac{|g-f|}{2} = \min(g, f)$$

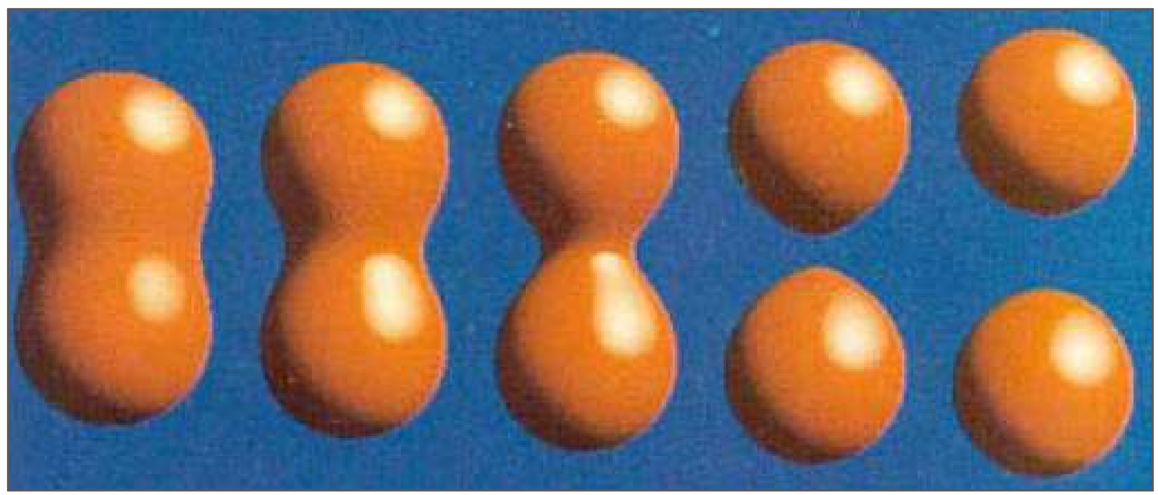
$$\lim_{\alpha \to 1} g \cap f = \tfrac{1}{2}\left(g + f + \sqrt{(g-f)^2}\right) = \frac{g+f}{2} + \frac{|g-f|}{2} = \max(g, f)$$

Implicit Curves and Surfaces

Blobs

- Suggested by Blinn [1982]
  - Defined implicitly by a potential function around a point $\mathbf{p}_i$: $g_i(\mathbf{p}) = a_i e^{-b_i \|\mathbf{p}-\mathbf{p}_i\|^2}$
  - Set operations by simple addition/subtraction





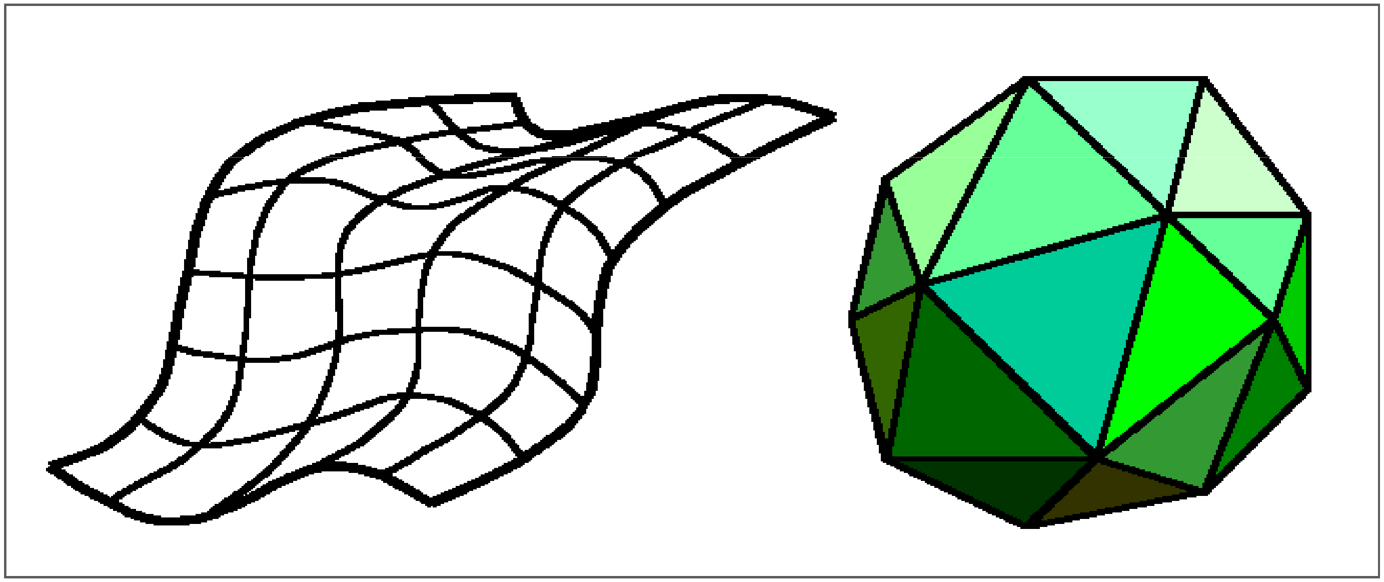
Implicit Curves and Surfaces

- Advantages
  - Easy to determine inside/outside
  - Easy to determine if a point is on the curve/surface
- Disadvantages
  - Hard to generate points on the curve/surface
  - Does not lend itself to (real-time) rendering

Polygonal Meshes

Polygonal Meshes

- Boundary representations of objects
  - Surfaces, polyhedrons, triangles, quadrilaterals



  - How are these objects stored?

Definitions

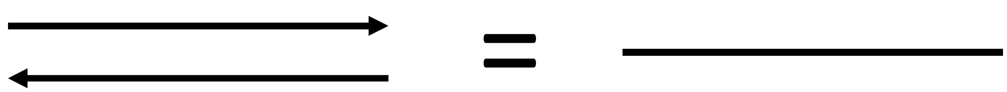
Geometric graph

- A Graph is a pair $G=(V,E)$
  - V is a nonempty set of n distinct vertices $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}$
  - E is a set of edges $(\mathbf{p}_i, \mathbf{p}_k)$
- If P is a (discrete) subset of R^d with $d \geq 2$, then $G=(V,E)$ is a *geometric graph*
- The *degree* or *valence* of a vertex describes the number of edges incident to this vertex

Definitions

Edges

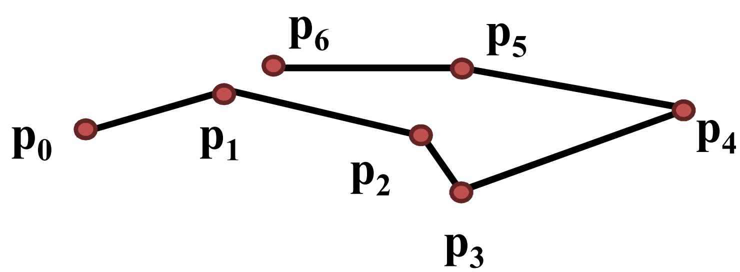
- Two edges are neighbors if they share a common vertex
- Edges are generally not oriented, and are noted as $(\mathbf{p}_i, \mathbf{p}_k)$
- Halfedges are edges with added orientation
- An edge is comprised of two halfedges



Definitions

Polygon

- A geometric graph Q=(V,E) with $V=\{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}\} \subset \mathrm{R}^d$ with $d \geq 2$ and $E=\{(\mathbf{p}_0, \mathbf{p}_1),(\mathbf{p}_1, \mathbf{p}_2),\ldots,(\mathbf{p}_{n-2}, \mathbf{p}_{n-1})\}$ is a *polygon*

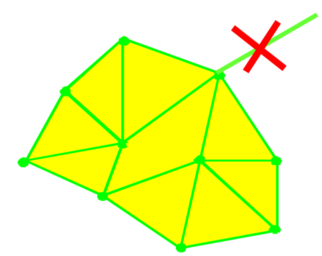


- A polygon is
  - Planar, if all vertices lie on a plane
  - Closed, if $\mathbf{p}_0 = \mathbf{p}_{n-1}$
  - Simple, if the polygon does not self-intersect

Definitions

Polygonal mesh

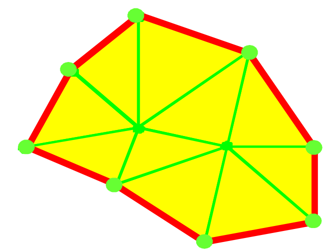
- A finite set M of closed, simple polygons $\mathrm{Q_i}$ is a polygonal mesh if:
  - The intersection of enclosed regions of any two polygons in M is empty
  - The intersection of two polygons in M is either empty, a vertex $\mathbf{v} \in \mathrm{V}$ or an edge $\mathbf{e} \in \mathrm{E}$
  - Every edge belongs to at least one polygon



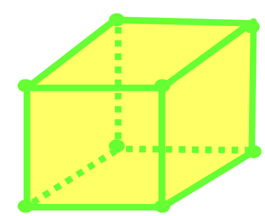
Definitions

Polygonal mesh

- (Continued) The set of all edges that belong to only one polygon is termed the *boundary* of the polygonal mesh, and is either empty or forms closed loops



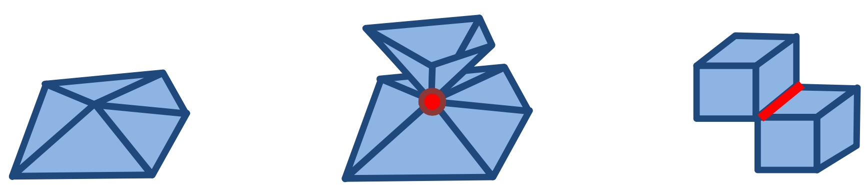
- If the set of edges that belong to only one polygon is empty, then the polygonal mesh is *closed*
- The set of all vertices and edges in a polygonal mesh form a graph



Definitions

Polyhedron

- A polygonal mesh is a polyhedron if
  - Each edge is part of two polygons (it is closed)
  - Every vertex $\mathbf{v} \in \mathrm{V}$ is part of finite, cyclic ordered set of polygons $\{\mathrm{Q}_\mathrm{i}\}$
    - The polygons incident to a vertex $\mathbf{v}$ can be ordered, such that Q_i and Q_j share an edge incident to $\mathbf{v}$



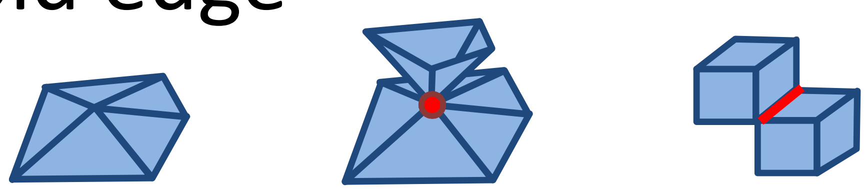
  - The union of all polygons forms a single connected component

Definitions

Manifold

- A polygonal mesh is a 2-manifold if It is everywhere locally homeomorphic to a (half) Euclidean 2-ball (a disk)
  - A coffee cup is homeomorphic to a torus

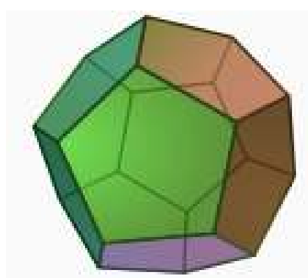
- Examples for a non-manifold vertex and a non-manifold edge



Definitions

Polyhedron

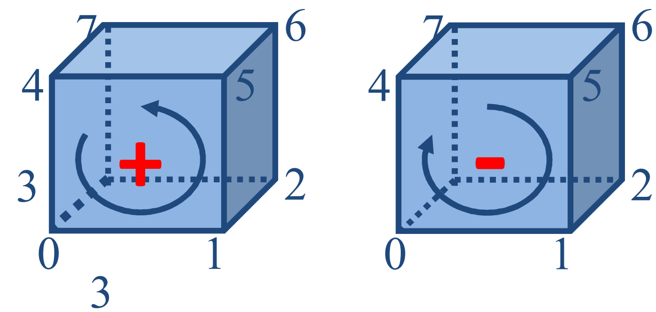
- The union of all polygonal areas is the *surface* of the polyhedron
- The polygonal areas of a polyhedron are also known as *faces*
- Every polyhedron partitions space into two areas; inside and outside the polyhedron



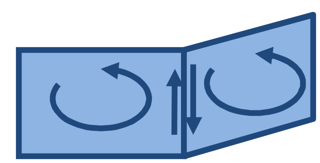
Definitions

Orientation

- Every face of a polygonal mesh is orientable
  - by defining “clockwise” (as opposed to “counterclockwise”). Two possible orientations
  - Defines the sign of the surface normal



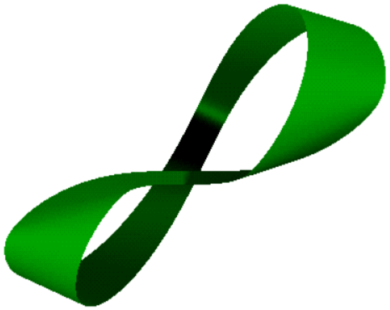
- Two neighboring facets are equally oriented, if the edge directions of the shared edge (induced by the face orientations) are opposing



Definitions

Orientability

- A polygonal mesh is orientable, if the incident faces to every edge can be equally oriented
  - If the faces are equally oriented for every edge, the mesh is *oriented*
- Notes
  - Every non-orientable closed mesh embedded in R^3 intersects itself
  - The surface of a polyhedron is always orientable

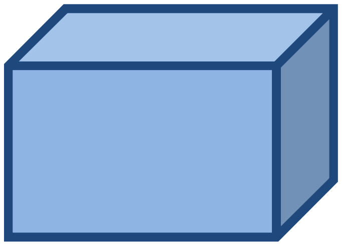


Euler-Poincaré Formula

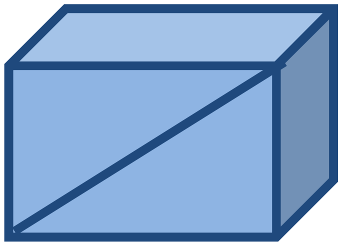
- Relation between #vertices, #edges and #faces of a polygonal mesh

- Example:

v = #vertices
e = #edges
f = #faces



v = 8
e = 12
f = 6



v = 8
e = 12+1
f = 6 +1

Euler-Poincaré Formula

- Theorem (Euler): The sum
$$\chi(M) = v - e + f$$
is constant for a given topology, no matter which mesh we choose
- If M has one boundary loop:
$$\chi(M) = v - e + f = 1$$
- If M is homeomorphic to a sphere:
$$\chi(M) = v - e + f = 2$$

Euler-Poincaré Formula

Usage

- Let‘s count the edges and faces in a closed triangle mesh:
  - Ratio of edges to faces: $e = 3/2\ f$
    - each edge belongs to exactly 2 triangles
    - each triangle has exactly 3 edges
  - Ratio of vertices to faces: $f \sim 2v$
    - $2 = v - e + f = v - 3/2\ f + f$
    - $2 - v = -f\ /\ 2$
  - Ratio of edges to vertices: $e \sim 3v$
  - Average degree of a vertex: 6
    - 2 vertices incident on each edge

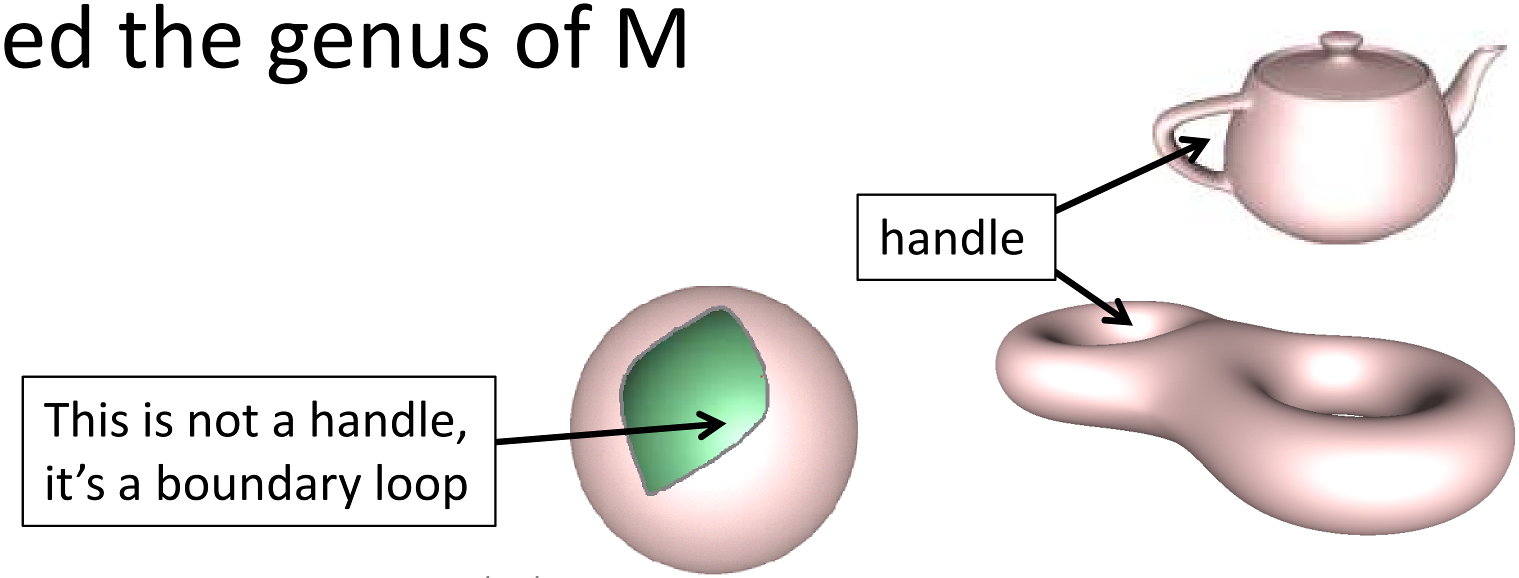
Euler-Poincaré Formula

Genus

- Theorem: if a polyhedron M is homeomorphic to a sphere with g handles (“holes“) then

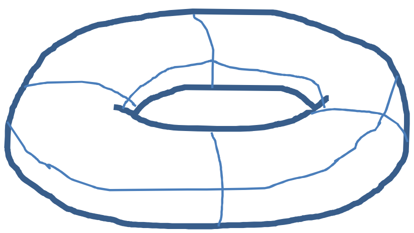
$$\chi(M) = v - e + f = 2(1 - g)$$

- g is called the genus of M



Euler-Poincaré Formula

Example: simple torus



$$v - e + f = 2(1 - g)$$

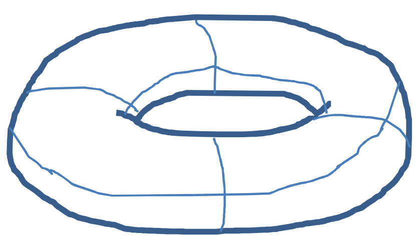
$$8 - 16 + 8 = 2(1 - 1)$$

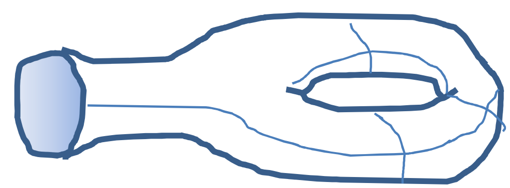
Euler-Poincaré Formula

Generalization

- Theorem: Let
  - v – # vertices
  - e – # edges
  - f – # faces
  - c – # connected components
  - h – # boundary loops
  - g – # handles (the genus)
    then:

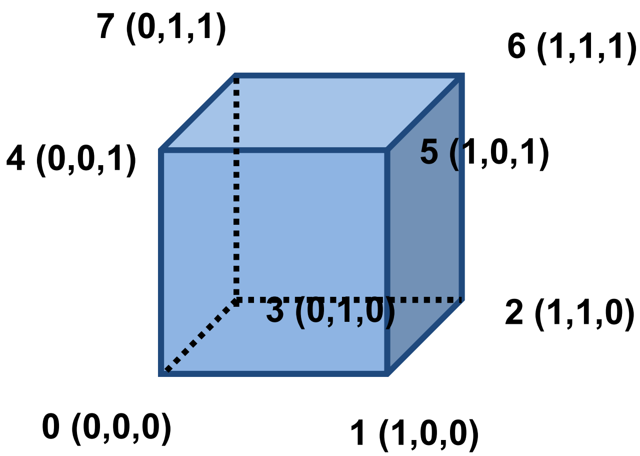
$$\boxed{v - e + f - h = 2\,(c - g)}$$





Data structures for meshes

Indexed Face Set



Vertex list (Coordinate3)

0	0.0	0.0	0.0
1	1.0	0.0	0.0
2	1.0	1.0	0.0
3	0.0	1.0	0.0
4	0.0	0.0	1.0
5	1.0	0.0	1.0
6	1.0	1.0	1.0
7	0.0	1.0	1.0

Face list (IndexedFaceSet)

0	0	1	2	3
1	0	1	5	4
2	1	2	6	5
3	2	3	7	6
4	3	0	4	7
5	4	5	6	7

Data structures for meshes

Space requirements

- Coordinates/attributes

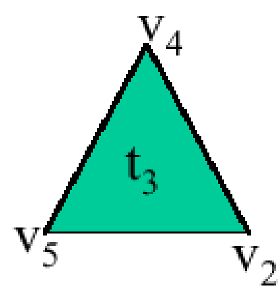
3x16+k bits/vertex

vertex 1	x	y	z	c
vertex 2	x	y	z	c
vertex 3	x	y	z	c

- Connectivity

$3x\log_2(V)$ bits/triangle

Triangle 1	1	2	3
Triangle 2	3	2	4
Triangle 3	4	2	5
Triangle 4	7	5	6
Triangle 5	6	5	8
Triangle 6	8	5	1

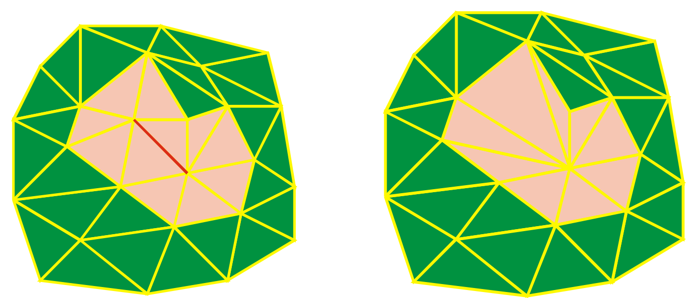


- When uncompressed, connectivity dominates
  - Reminder: f = 2v... so after 256 vertices

Data structures for meshes

Indexed Face Set – Problems

- Information about neighbors is not explicit
  - Finding neighboring vertices/edges/faces etc. costs O(v) time!
  - Local mesh modifications cost O(v)

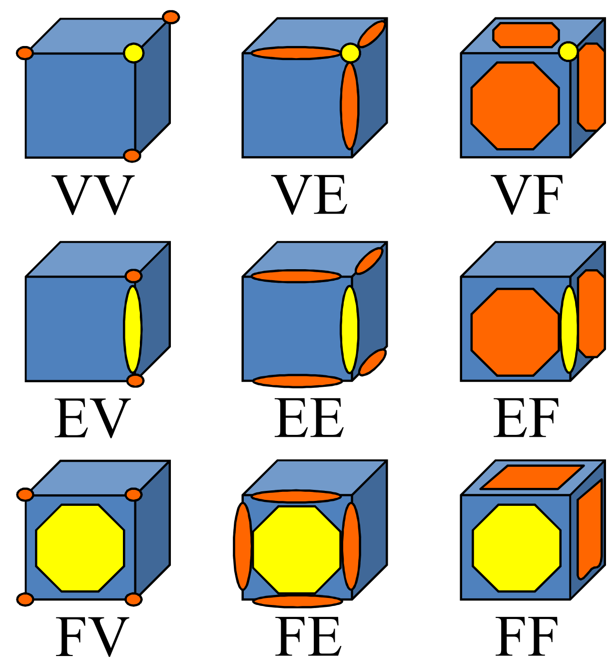


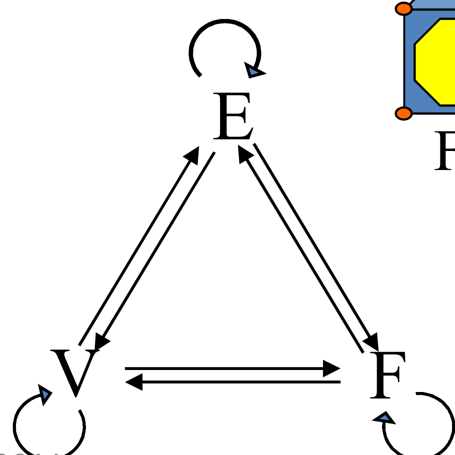
  - Breadth-first search costs O(k*v) where k = # found vertices

Data structures for meshes

Neighborhood relations [Weiler 1985]

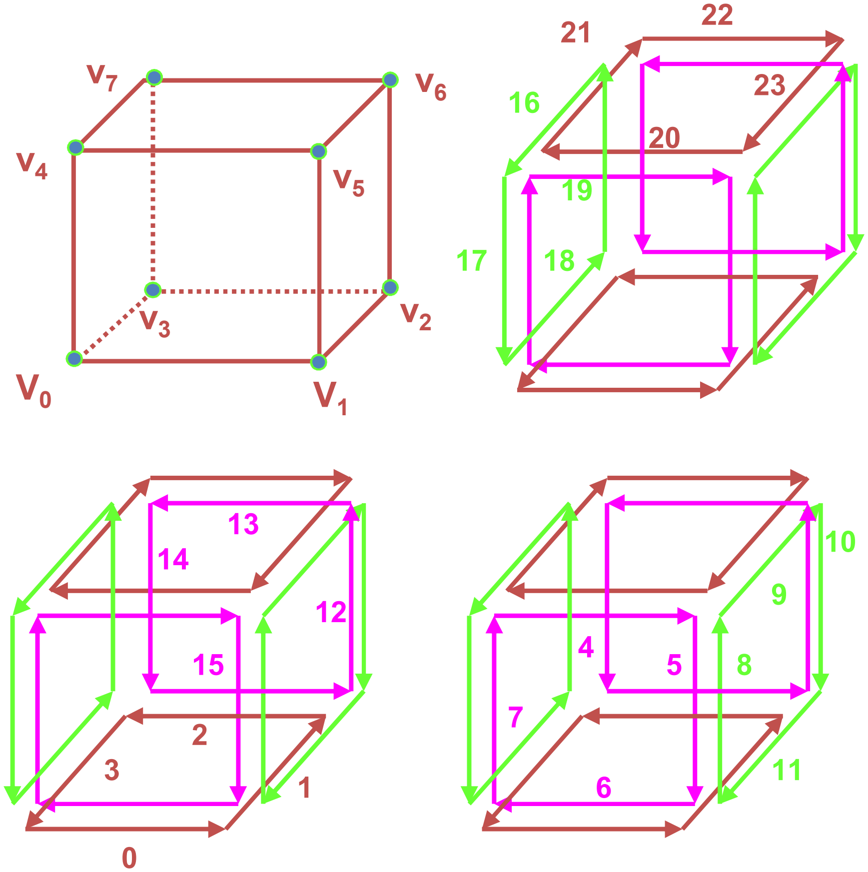
- All possible neighborhood relationships:
  1. Vertex – Vertex VV
  2. Vertex – Edge VE
  3. Vertex – Face VF
  4. Edge – Vertex EV
  5. Edge – Edge EE
  6. Edge – Face EF
  7. Face – Vertex FV
  8. Face – Edge FE
  9. Face – Face FF





Data structures for meshes

Half-edge data structure



Vertexlist

v	coord			he
0	0.0	0.0	0.0	0
1	1.0	0.0	0.0	1
2	1.0	1.0	0.0	2
3	0.0	1.0	0.0	3
4	0.0	0.0	1.0	4
5	1.0	0.0	1.0	9
6	1.0	1.0	1.0	13
7	0.0	1.0	1.0	16

Face

f	e
0	e0
1	e8
2	e4
3	e16
4	e12
5	e20

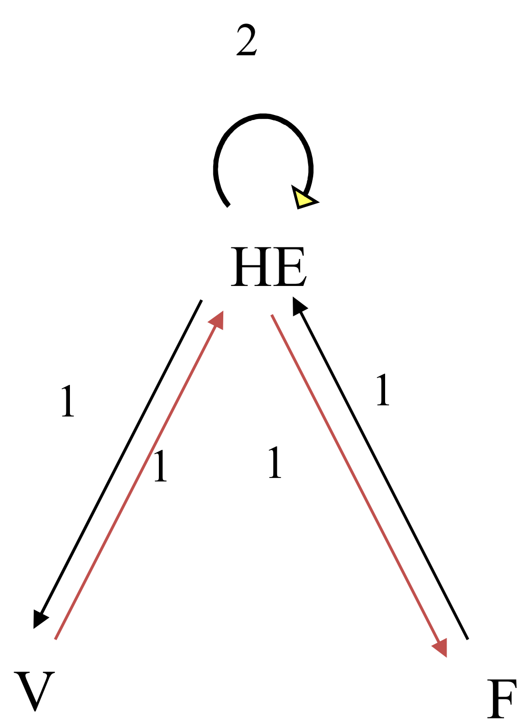
Half-Edgelist

he	vstart	next	prev	opp	he	vstart	next	prev	opp
0	0	1	3	6	12	2	13	15	10
1	1	2	0	11	13	6	14	12	22
2	2	3	1	15	14	7	15	13	19
3	3	0	2	18	15	3	12	14	2
4	4	5	7	20	16	7	17	19	21
5	5	6	4	8	17	4	18	16	7
6	1	7	5	0	18	0	19	17	3
7	0	4	6	17	19	3	16	18	14
8	1	9	11	5	20	5	21	23	4
9	5	10	8	23	21	4	22	20	16
10	6	11	9	12	22	7	23	21	13
11	2	8	10	1	23	6	20	22	9

Data structures for meshes

Half-edge data structure

- Each atomic insertion into the data structure (i.e., vertex, edge or face insertion) requires constant space and time

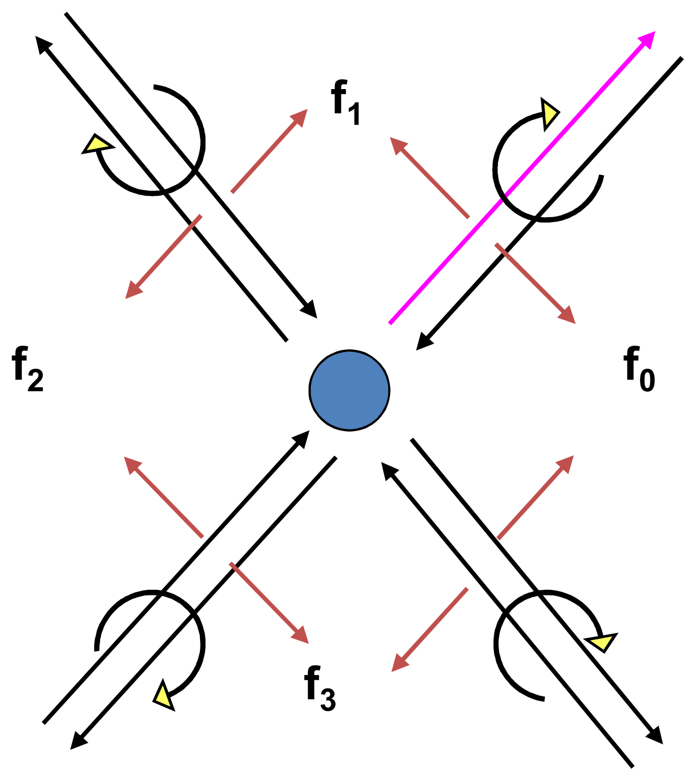


Triangle face:
prev = next->next

Data structures for meshes

Half-edge data structure

- All basic queries take constant O(1) time!
  - In particular, the query time is independent of the model size



Data structures for meshes

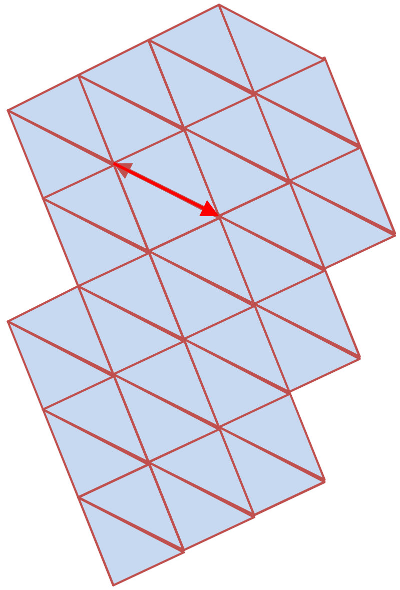
Half-edge data structure

- Example: efficient breadth-first search

```
//q: Queue (FIFO) of HalfEdges

HalfEdge he;
q.append(he);
if (he.opposite != null)
   q.append(he.opposite);

while (! q.isEmpty()) {
  he=q.first();
  // do work
  if (he.next.opposite != null)
    q.append(he.next.opposite);
  if (he.next.next.opposite != null)
    q.append(he.next.next.opposite)
}
```



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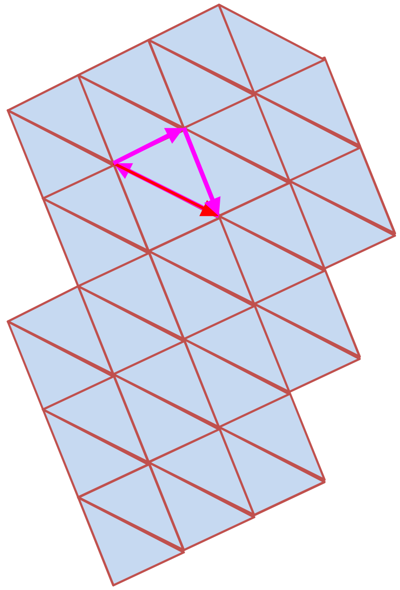

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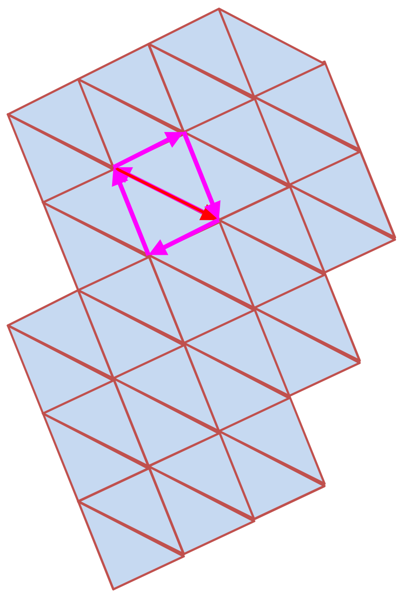

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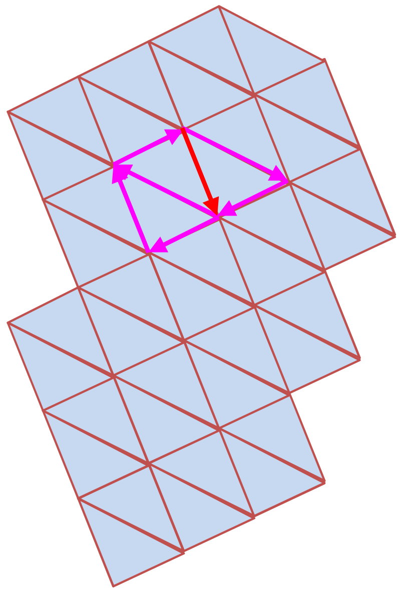

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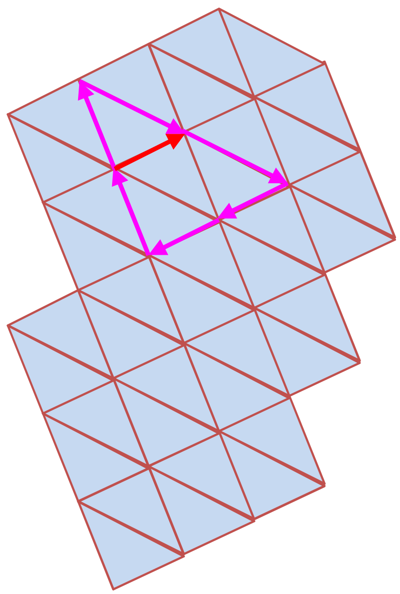

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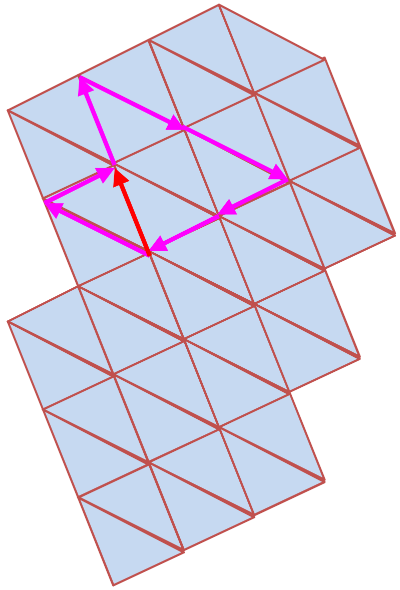

Data structures for meshes

Criteria for design

- Maximal number of vertices (i.e., how large are the models?)
- Available memory size
- Required operations
  - Mesh updates (edge collapse, edge flip)
  - Neighborhood queries
- Distribution of operations (what are the most common/frequent ones?)
- How can we compare different data structures?