CS 523: Computer Graphics, Spring 2011

Shape Modeling

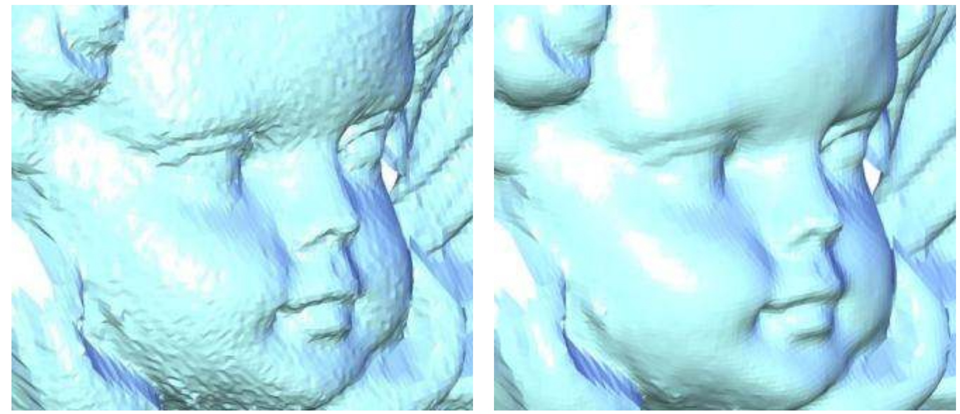
Differential Geometry of Surfaces

Differential Geometry of Surfaces

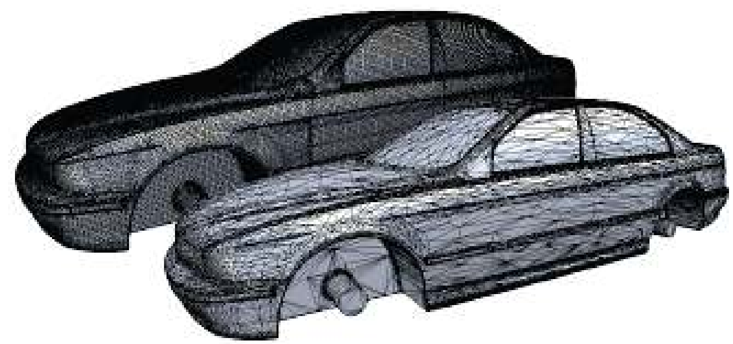
Continuous and Discrete

Motivation

- Smoothness
  - Mesh smoothing



- Adaptive tessellation
  - Mesh decimation



- Shape preserving mesh editing

Surfaces

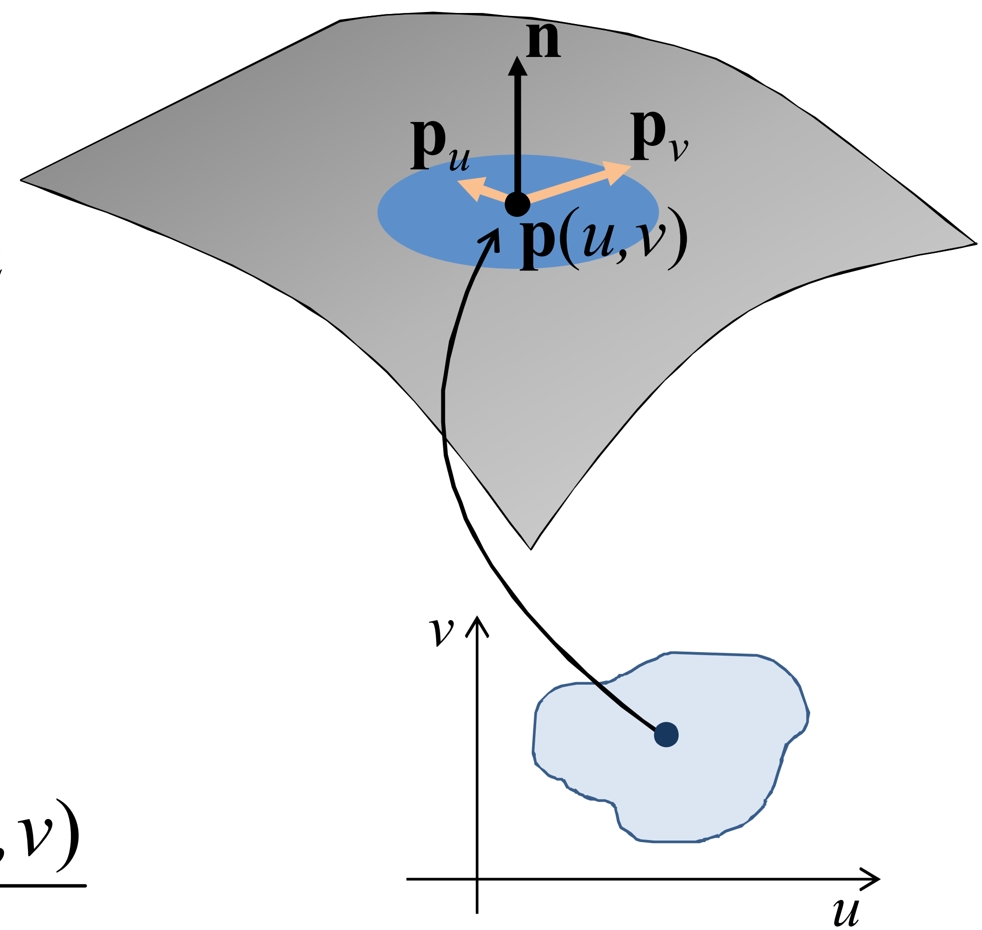
Parametric form

- Continuous surface

$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \Re^2$$

- Tangent plane at point $\mathbf{p}(u,v)$ is spanned by

$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \qquad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$



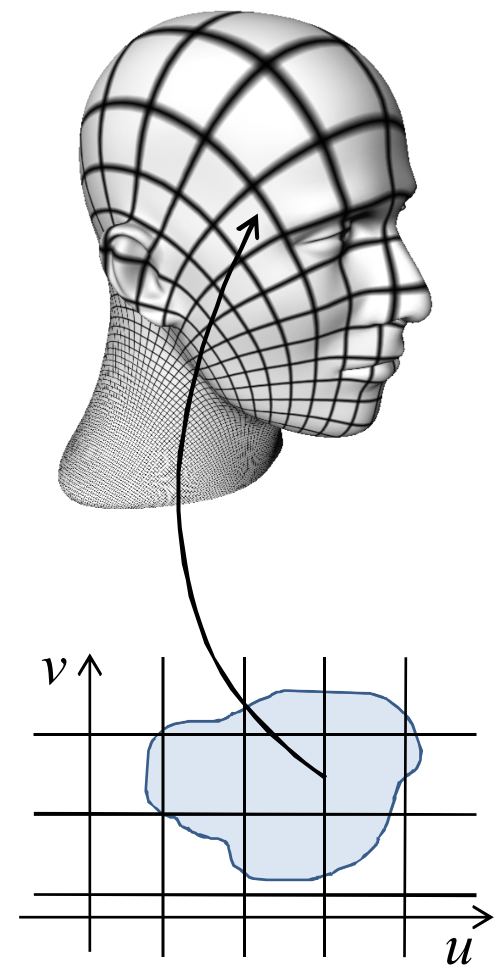
Surfaces

Isoparametric lines

- Lines on the surface when keeping one parameter fixed

$$\gamma_{u_0}(v) = \mathbf{p}(u_0, v)$$

$$\gamma_{v_0}(u) = \mathbf{p}(u, v_0)$$



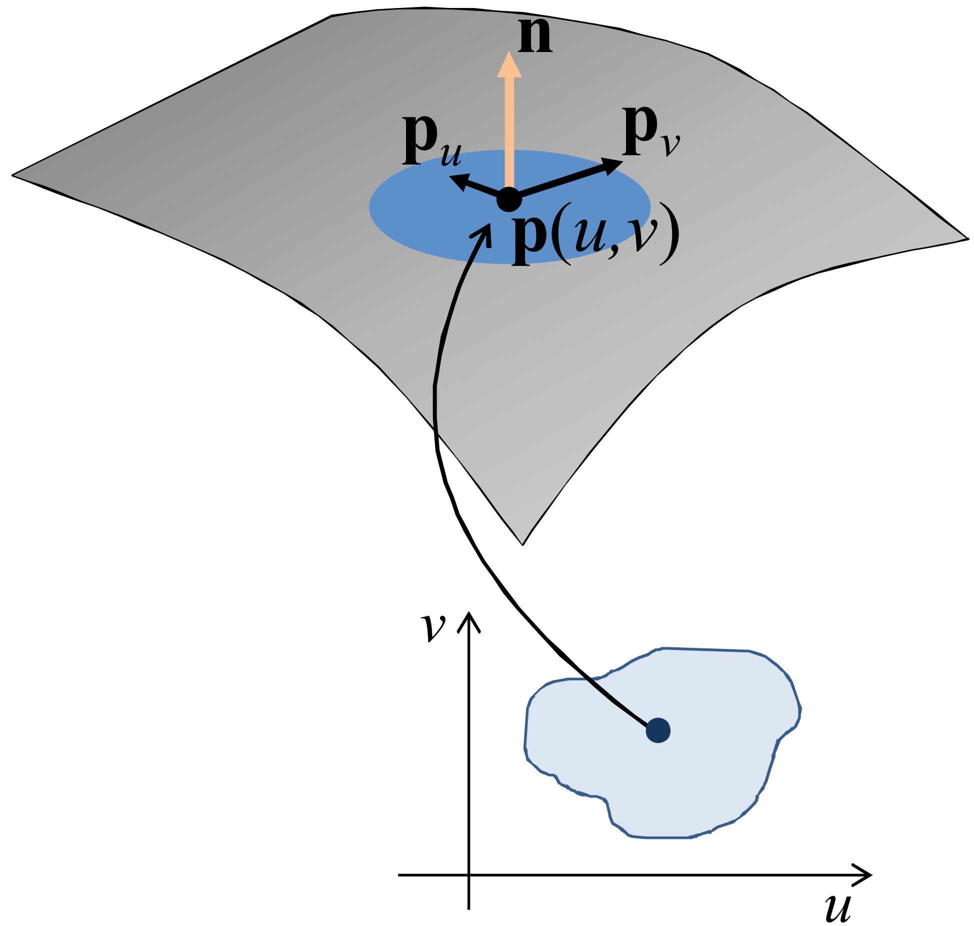
Surfaces

- Surface normal:

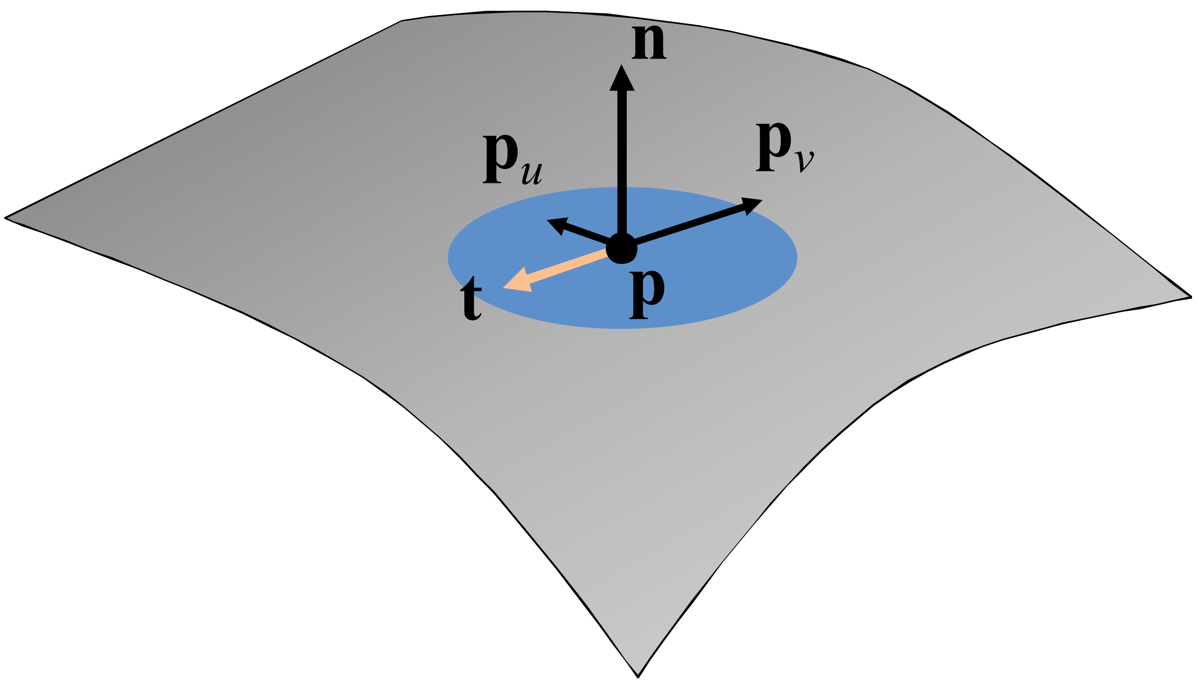
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\left\|\mathbf{p}_u \times \mathbf{p}_v\right\|}$$

  - Assuming *regular* parameterization, i.e.,

$$\mathbf{p}_u \times \mathbf{p}_v \neq 0$$



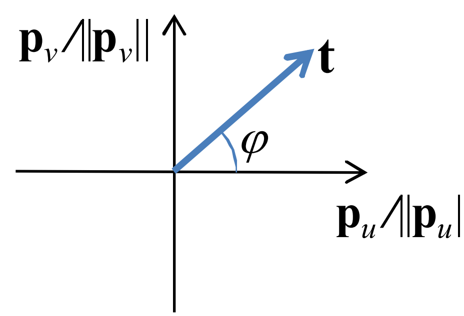
Normal curvature



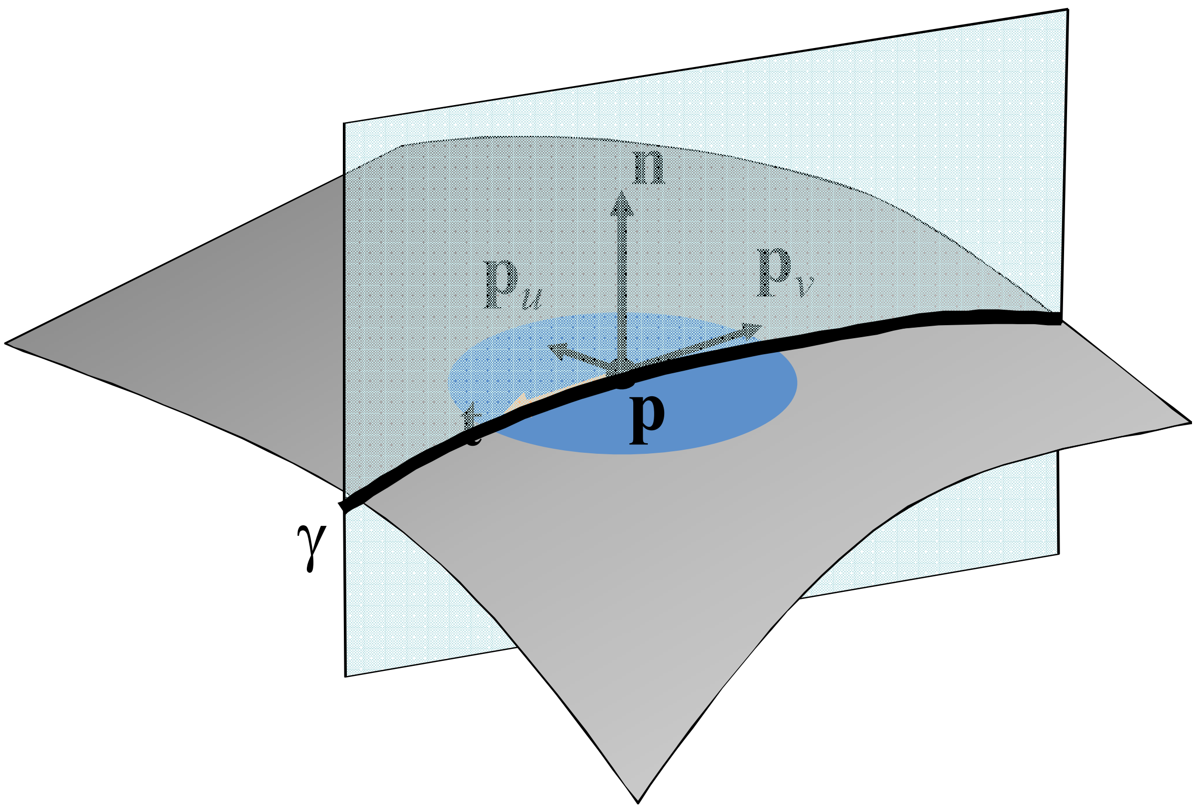
$$\mathbf{n} = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\left\| \mathbf{p}_u \times \mathbf{p}_v \right\|}$$

Direction **t** in the tangent plane:

$$\mathbf{t} = \cos\varphi \frac{\mathbf{p}_u}{\left\|\mathbf{p}_u\right\|} + \sin\varphi \frac{\mathbf{p}_v}{\left\|\mathbf{p}_v\right\|}$$



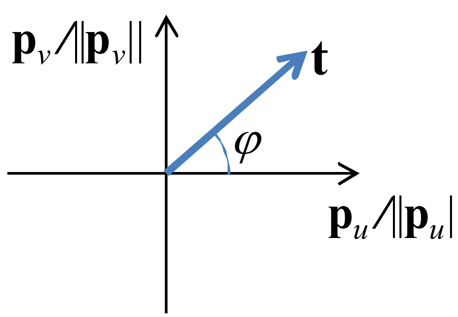
Normal curvature



The curve γ is the intersection of the surface with the plane through $\mathbf{n}$ and $\mathbf{t}$.

Normal curvature:

$$\kappa(\gamma(\mathbf{p}))$$



Surface curvatures

- Principal curvatures
  - Maximal curvature $\kappa_1 = \max_{\phi} \kappa_n(\phi)$
  - Minimal curvature $\kappa_2 = \min_{\phi} \kappa_n(\phi)$
- Mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\phi) d\phi$$

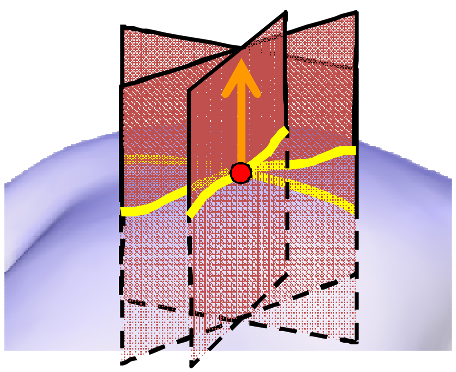
- Gaussian curvature

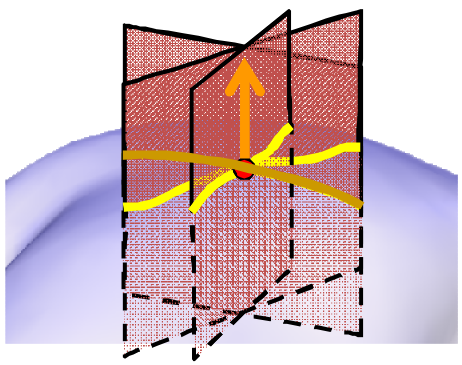
$$K = \kappa_1 \cdot \kappa_2$$

Mean curvature

- Intuition for mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\phi)d\phi$$





Surface curvatures

$$\kappa_1 = \max_{\phi} \kappa_n(\phi)$$

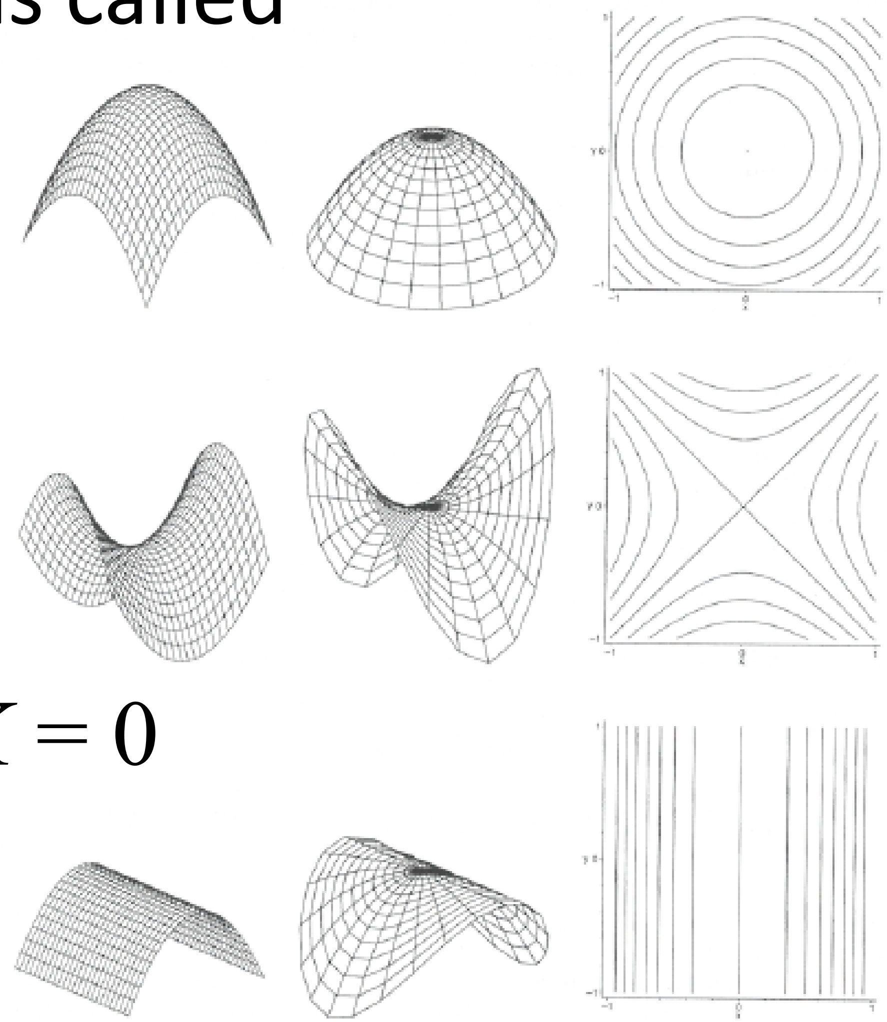
$$\kappa_2 = \min_{\phi} \kappa_n(\phi)$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)$$

$$K = \kappa_1 \cdot \kappa_2$$

Classification

- A point $\mathbf{p}$ on the surface is called
  - Elliptic, if $K > 0$
  - Parabolic, if $K = 0$
  - Hyperbolic, if $K < 0$
  - Umbilical, if $\kappa_1 = \kappa_2$
- Developable surface $\Leftrightarrow K = 0$



Laplace operator

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}.$$

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami operator

- Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$$

function on manifold $\mathcal{S}$

divergence operator

Laplace-Beltrami operator

- Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \mathbf{x} = \mathrm{div}_{\mathcal{S}} \, \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

coordinate function

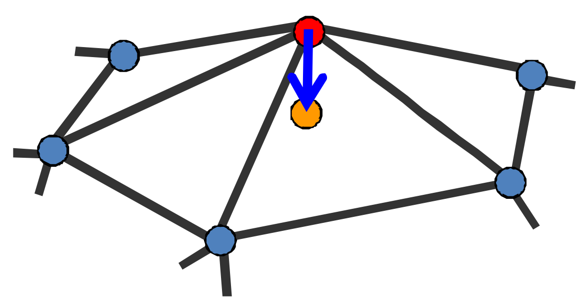
divergence operator

surface normal

Discrete differential operators

- Assumption: meshes are piecewise linear approximations of smooth surfaces

- Approach: approximate differential properties at point $\mathbf{x}$ as spatial average over local mesh neighborhood $N(\mathbf{x})$ where typically
  - $\mathbf{x}$ = mesh vertex
  - $N_k(\mathbf{x})$ = k-ring neighborhood or local geodesic ball



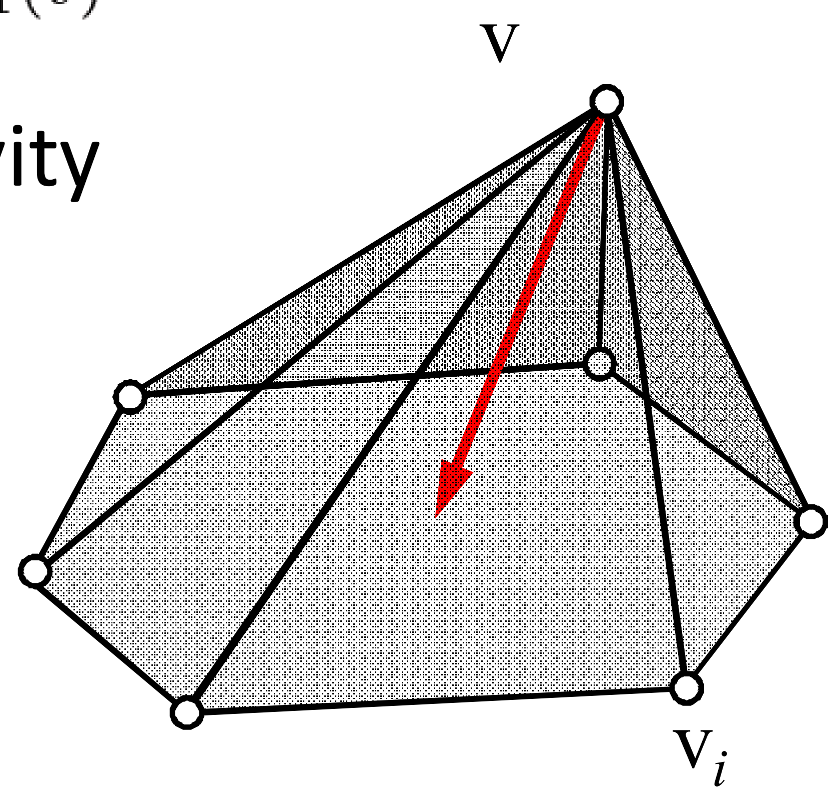
Discrete Laplace-Beltrami

- Uniform discretization - $L(v)$ or Δv

$$\Delta_{uni} f(v) := \frac{1}{|\mathcal{N}_1(v)|} \sum_{v_i \in \mathcal{N}_1(v)} \left(f(v_i) - f(v) \right)$$

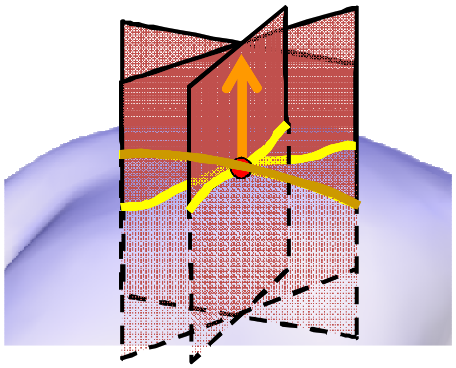
- Depends only on connectivity = simple and efficient

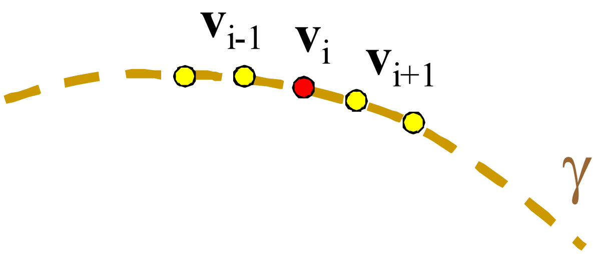
- Bad approximation for irregular triangulations



Discrete Laplace-Beltrami

- Intuition for uniform discretization





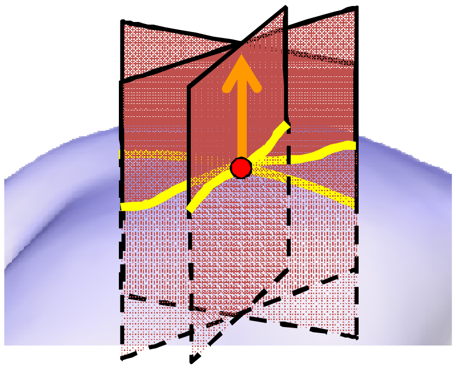
$$H = \int_0^{2\pi} \kappa(\theta)\, d\theta$$

$$\kappa = \left\| \ddot{\gamma} \right\|$$

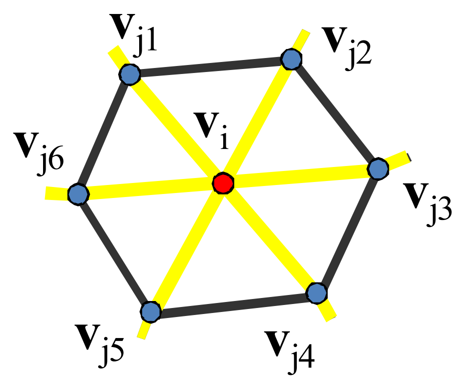
$$\ddot{\boldsymbol{\gamma}} \approx \left(\mathbf{v}_{i-1} - \mathbf{v}_i\right) - \left(\mathbf{v}_i - \mathbf{v}_{i+1}\right) = \mathbf{v}_{i-1} + \mathbf{v}_{i+1} - 2\mathbf{v}_i$$

Discrete Laplace-Beltrami

- Intuition for uniform discretization



$$H = \int_0^{2\pi} \kappa(\theta)\, d\theta$$

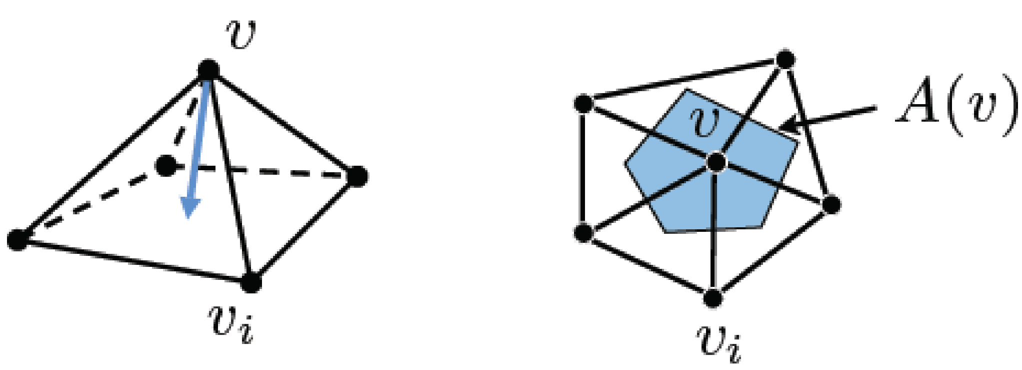


$$\begin{aligned} &\mathbf{v}_{j1} + \mathbf{v}_{j4} - 2\mathbf{v}_i \quad + \\ &\mathbf{v}_{j2} + \mathbf{v}_{j5} - 2\mathbf{v}_i \quad + \\ &\mathbf{v}_{j3} + \mathbf{v}_{j6} - 2\mathbf{v}_i \quad = \\ &= \sum_{k=1}^{6} \mathbf{v}_{j_k} - 6\mathbf{v}_i = -L(\mathbf{v}_i) \end{aligned}$$

Discrete Laplace-Beltrami

- Cotangent formula

$$\Delta_{\mathcal{S}} f(v) := \frac{1}{2A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i) \left(f(v_i) - f(v) \right)$$

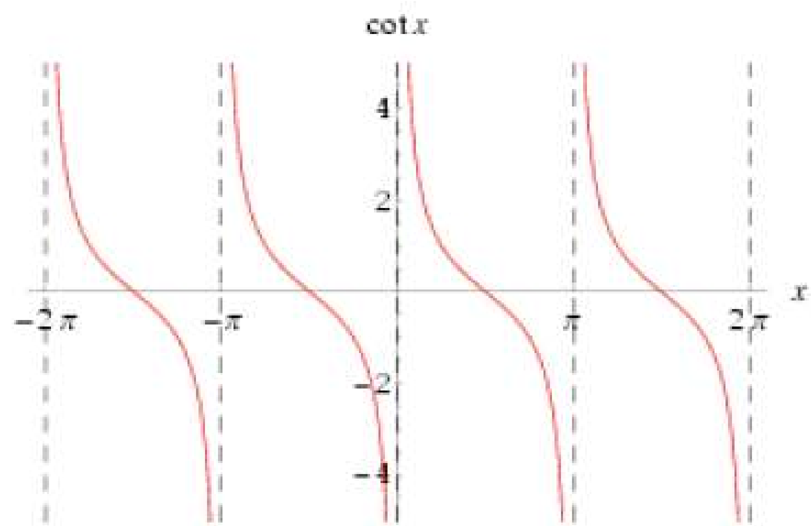


Discrete Laplace-Beltrami

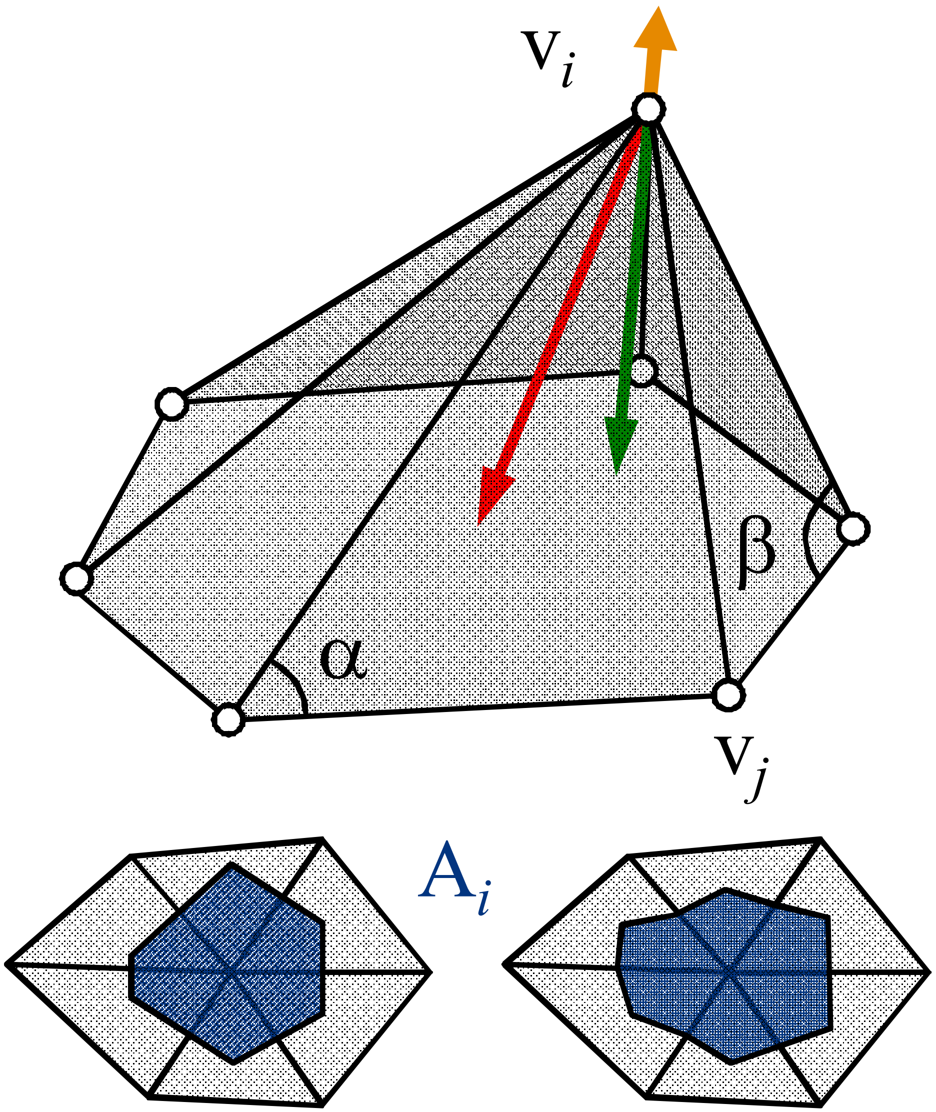
- Cotangent formula

$$\Delta_S f(v) := \frac{1}{2A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i) (f(v_i) - f(v))$$

- Problems
  - Potentially negative weights
  - Depends on geometry

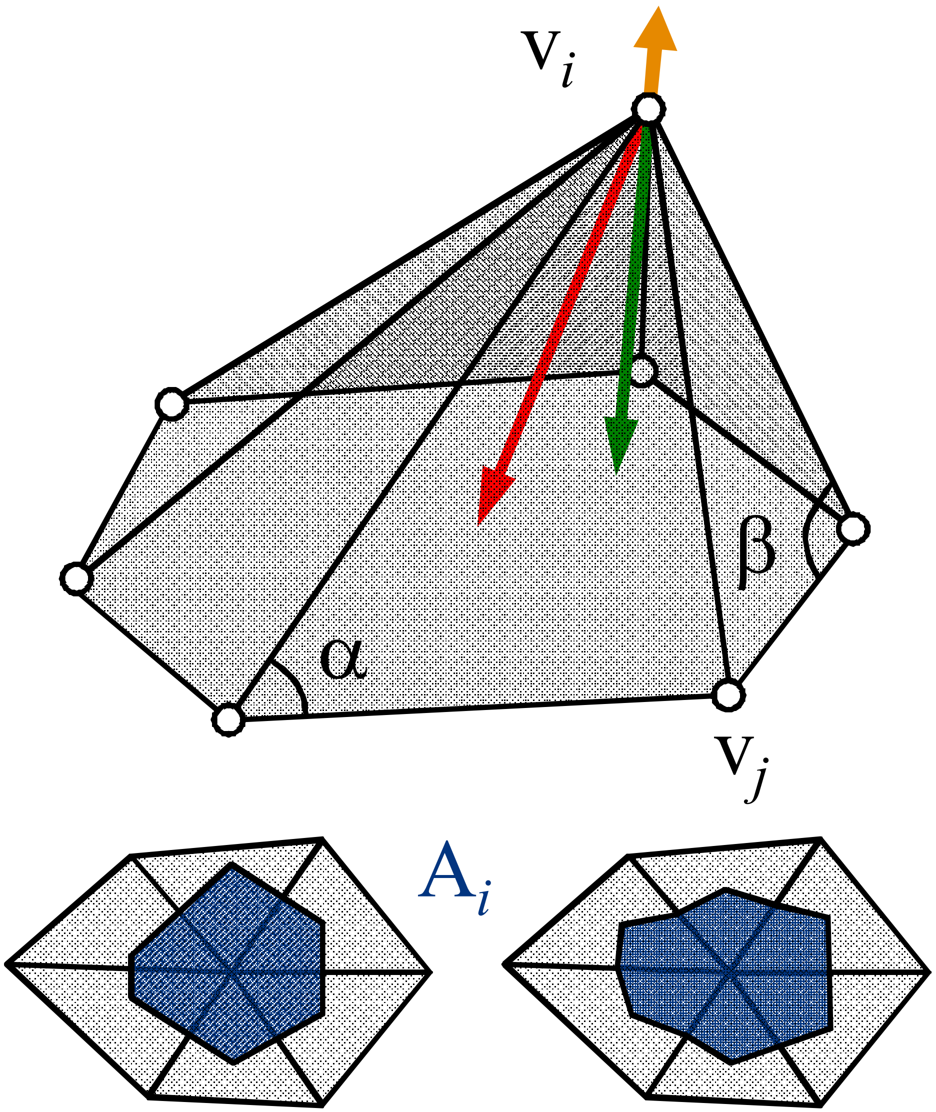


Discrete Laplace-Beltrami



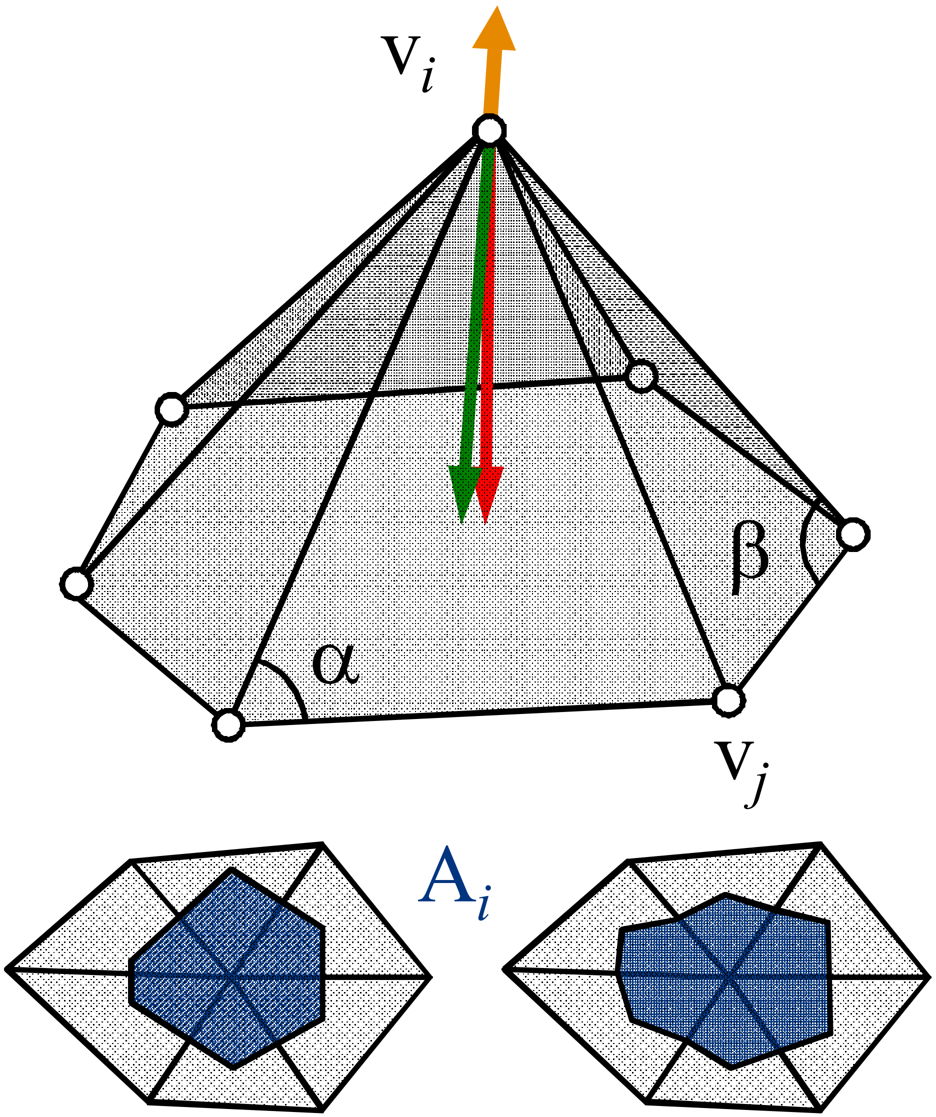
- Laplacian operators
  - Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
  - Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
  - Mean curvature normal

Discrete Laplace-Beltrami



- Laplacian operators
  - Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
  - Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
  - Mean curvature normal
- **Cotangent Laplacian** = **mean curvature normal** x **vertex area ($\mathbf{A}_i$)**
- **For nearly equal edge lengths Uniform ≈ Cotangent**

Discrete Laplace-Beltrami



- Laplacian operators
  - Uniform Laplacian $\mathbf{L}_u(\mathbf{v}_i)$
  - Cotangent Laplacian $\mathbf{L}_c(\mathbf{v}_i)$
  - Mean curvature normal
- Cotangent Laplacian = mean curvature normal x vertex area ($\mathbf{A}_i$)
- **For nearly equal edge lengths**
  Uniform ≈ Cotangent

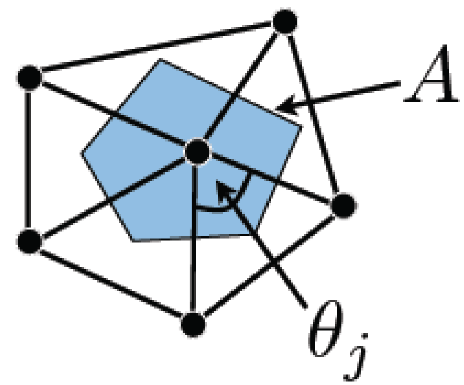
Discrete curvatures

- Mean curvature

$$H = \|\Delta_{\mathcal{S}} \mathbf{x}\|$$

- Gaussian curvature

$$G = (2\pi - \sum_j \theta_j)/A$$

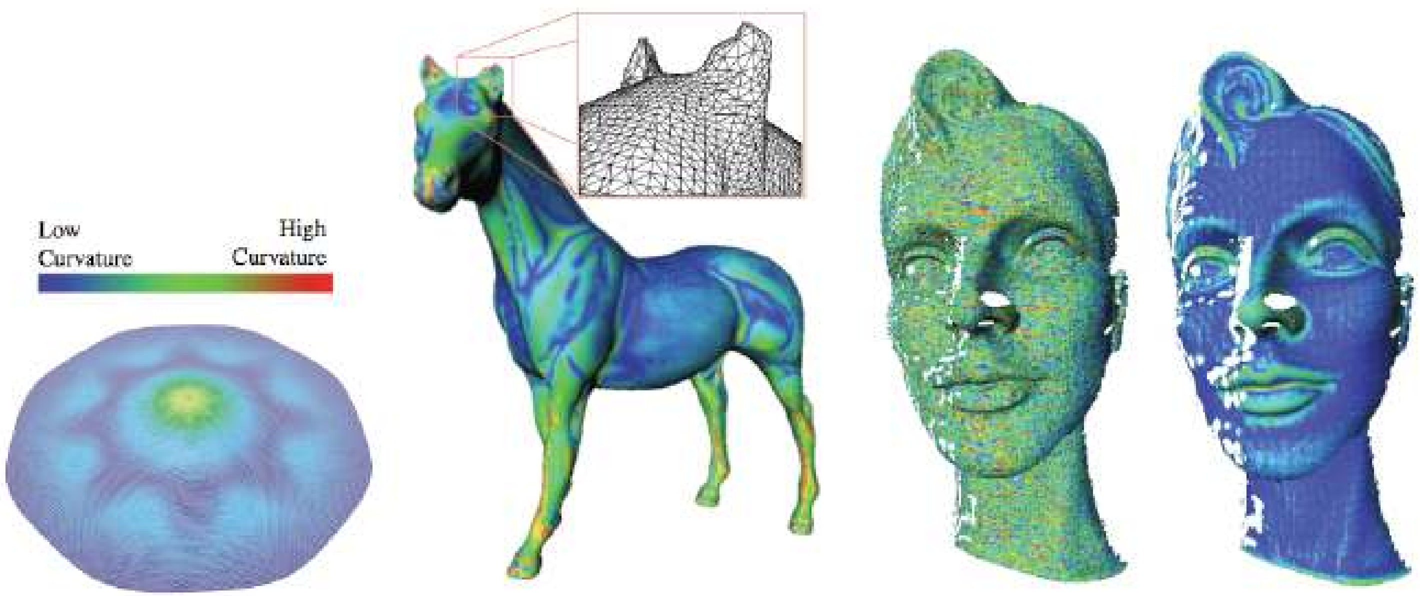


- Principal curvatures

$$\kappa_1 = H + \sqrt{H^2 - G} \qquad \kappa_2 = H - \sqrt{H^2 - G}$$

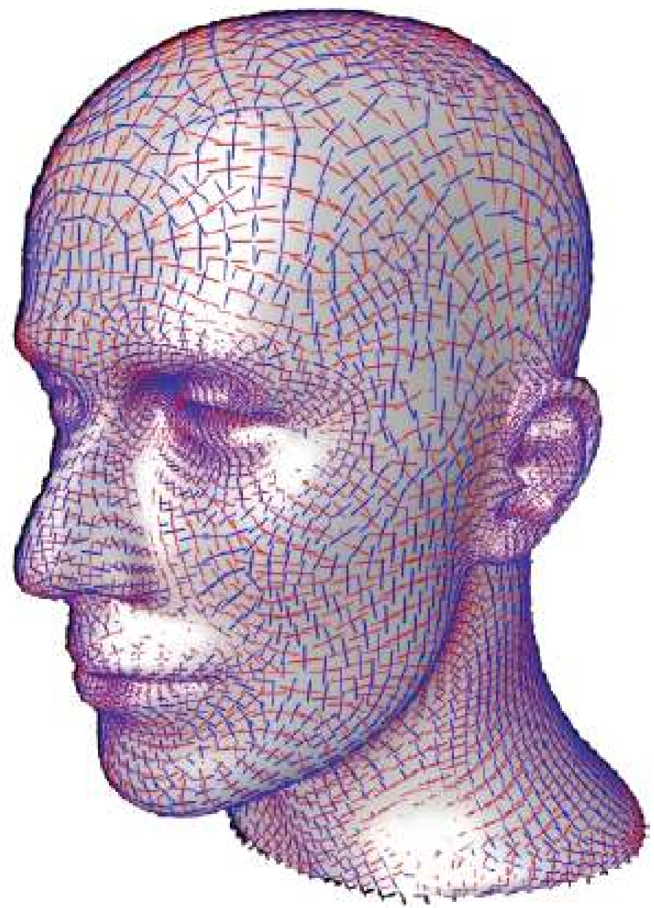
Links and literature

- M. Meyer, M. Desbrun, P. Schroeder, A. Barr *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds*, VisMath, 2002



Links and literature

- P. Alliez, *Estimating Curvature Tensors on Triangle Meshes*, Source Code
  - http://www-sop.inria.fr/geometrica/team/Pierre.Alliez/demos/curvature/



principal directions

Links and literature

- Grinspun et al.: *Computing discrete shape operators on general meshes, Eurographics 2006*